Theory of Microlensing and Planetary Microlensing:
Basic Concepts

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Outline

- What is microlensing?
- Lens equation, images, and magnifications
- Binary lens equations
- Critical curves, caustics and planetary microlensing
- Summary
- Suggested readings

What is Galactic microlensing?

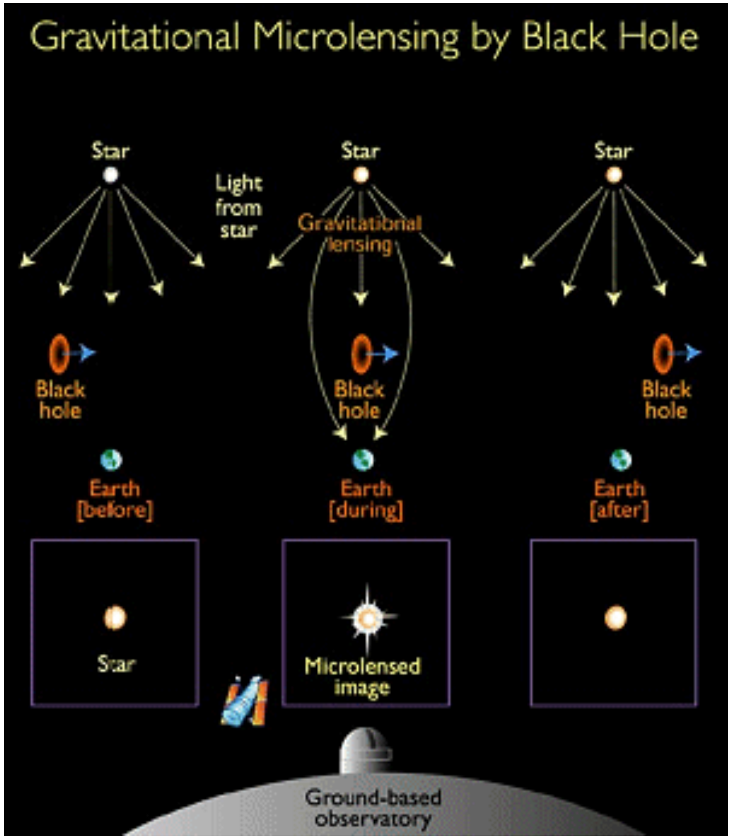
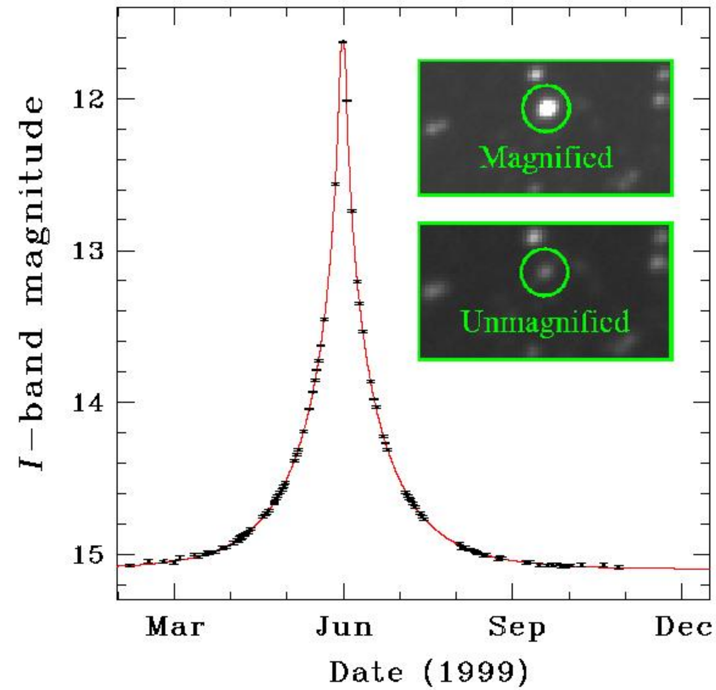


Image credit: NASA/ESA



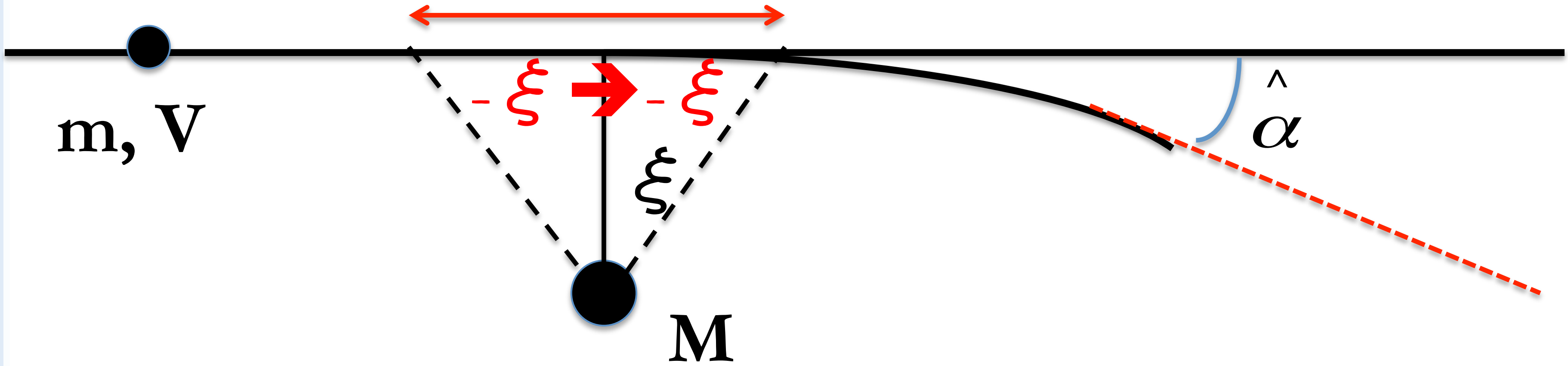
Light curve is symmetric, achromatic & non-repeating (Paczynski 1986)

Image separation is too small to resolve, we observe magnification effects.

Microlensing can be used to

- Discover extrasolar planets (~15)
- Study MW structure, stellar mass black holes, …

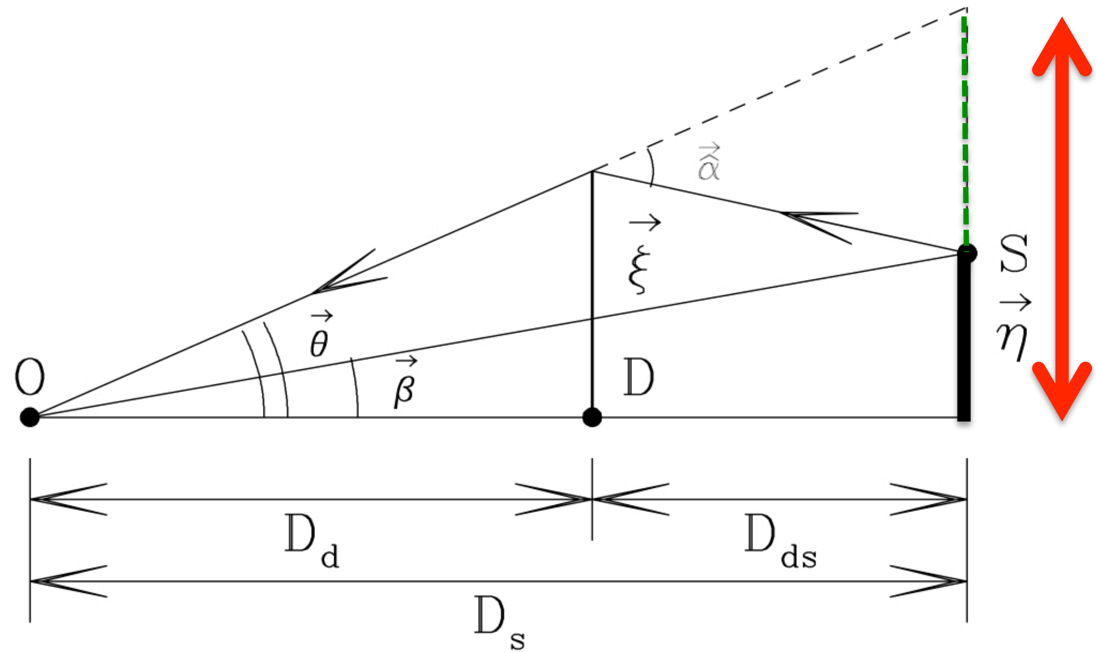
Deflection angle: impulse approximation



$$\left|\vec{\hat{\alpha}}\right| \approx \frac{mV_\perp}{mV} = \frac{F_\perp \Delta t}{mV} = \frac{(GMm/\xi^2)\,(2\xi/V)}{mV} = \frac{2GM}{V^2\xi}$$

In general relativity, $\left|\vec{\hat{\alpha}}\right| = \frac{2GM}{c^2\xi}$ **x2**

Lens equation



$$\vec{\eta} + D_{ds}\vec{\hat{\alpha}} = \frac{D_s}{D_d}\vec{\xi}, \quad \left|\vec{\hat{\alpha}}\right| = \frac{4GM}{c^2\xi} \quad \xrightarrow{\div D_s} \quad \frac{\vec{\eta}}{D_s} + \frac{D_{ds}}{D_s}\vec{\hat{\alpha}} = \frac{\vec{\xi}}{D_d}$$

$$\vec{\beta} + \vec{\alpha} = \vec{\theta}, \quad \left|\vec{\alpha}\right| = \frac{4GM}{c^2 D_d \theta}\frac{D_{ds}}{D_s} = \frac{\theta_E^2}{\theta}$$

$$\boxed{\theta_E^2 = \frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}}$$

Single point lens equation

$$\vec{\beta} + \vec{\alpha} = \vec{\theta}, \qquad \vec{\alpha} = \frac{4GM}{c^2 D_d \theta} \frac{D_{ds}}{D_s} \frac{\vec{\theta}}{\theta} = \frac{\theta_E^2}{\theta} \frac{\vec{\theta}}{\theta}$$

$$\vec{\beta} = \vec{\theta} - \frac{\theta_E^2}{\theta} \frac{\vec{\theta}}{\theta}$$

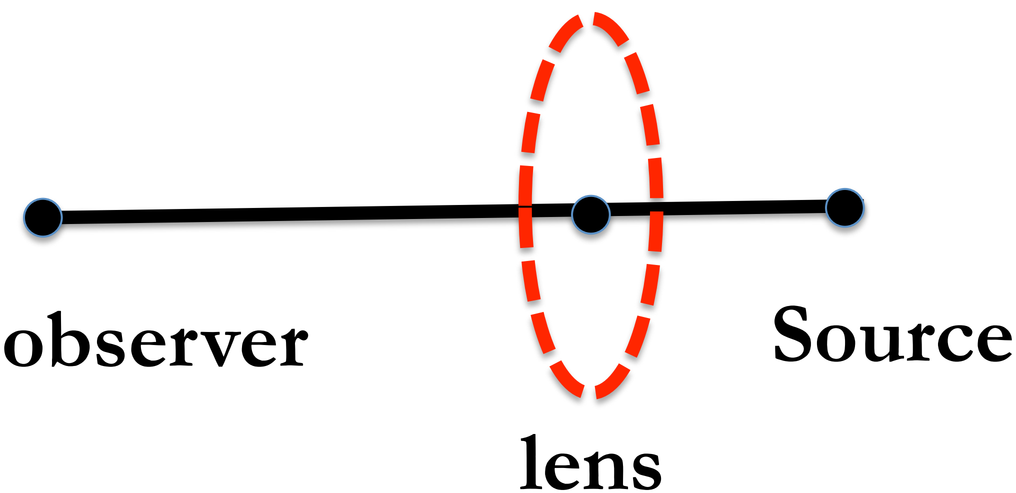
Setting the angular Einstein radius to unity, we have

$$\beta = \theta - \frac{1}{\theta}$$

Lens images

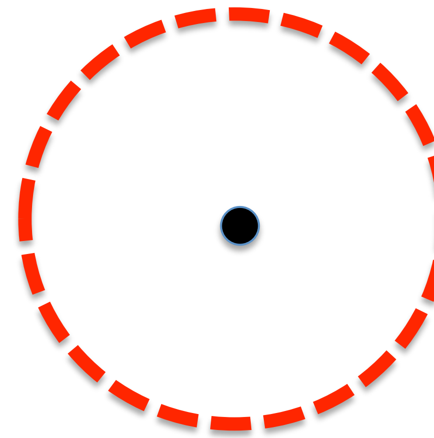
Lens equation: $\beta = \theta - \frac{1}{\theta}$

If $\beta = 0$, **Solution:** $\theta = \pm 1$



Side view

Einstein Ring



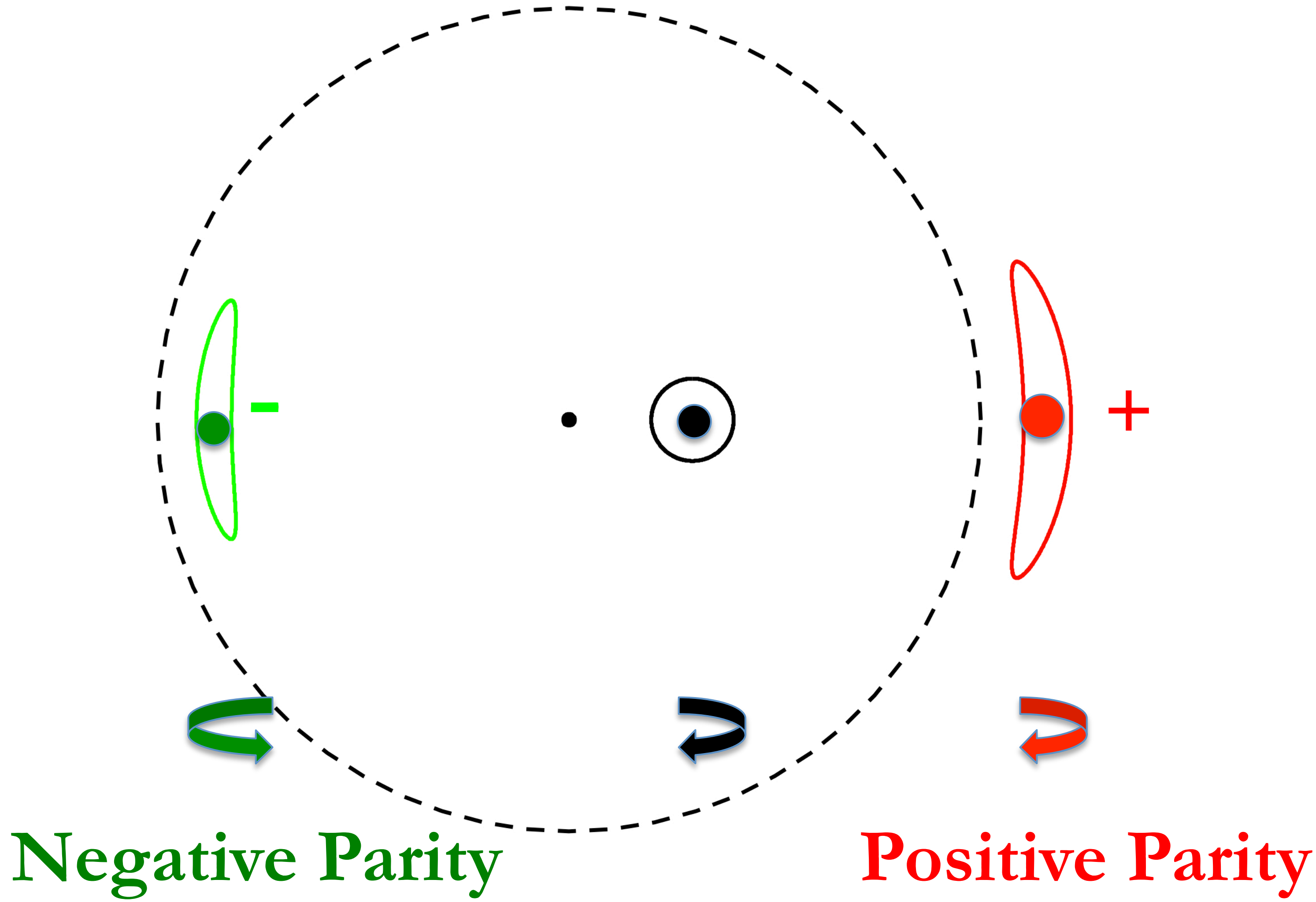
Plane of Sky

Lens images

Lens equation: $\beta = \theta - \frac{1}{\theta}$

If $\beta \neq 0$, **Solutions:** $\theta_{\pm} = \frac{\beta \pm \sqrt{\beta^2 + 4}}{2}$

Lens images



Lens mapping

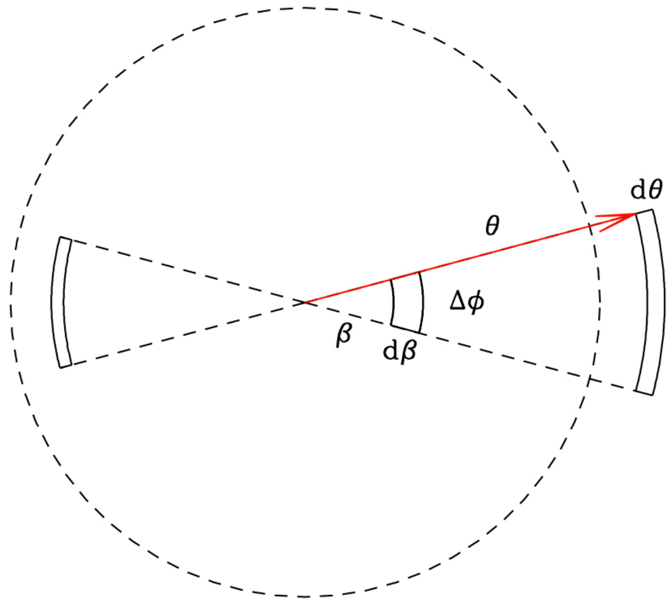
- Lens equation $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$ is a 2D mapping between the source plane to the image plane

$$\vec{\beta} \rightarrow \vec{\theta}$$

  – The mapping may not be unique (multiple images)

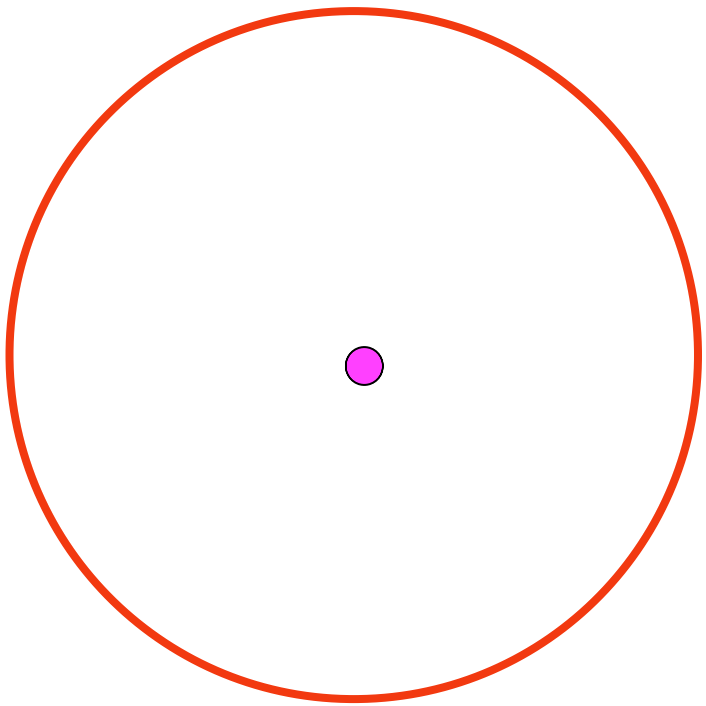
- Gravitational lensing conserves surface brightness
  – Magnification is just the ratio of the image area to the source area
  – In general, $$\mu = \det\left(\frac{\partial^2 \vec{\theta}}{\partial \vec{\beta}^2}\right) = J^{-1}, \ J = \det\left(\frac{\partial^2 \vec{\beta}}{\partial \vec{\theta}^2}\right)$$

Single lens magnification



- Magnification $$\mu = \frac{\theta\Delta\Phi\, d\theta}{\beta\Delta\Phi\, d\beta} = \frac{\theta}{\beta}\frac{d\theta}{d\beta}$$
- $|\mu_+| - |\mu_-| = 1$ $(\mu_+ > 0,\ \mu_- > 0)$
- $\mu_{total} = |\mu_+| + |\mu_-| = (\beta^2+2)/(\beta(\beta^2+4)^{1/2})$

critical curves and caustics: point lens



- Caustics: point source positions with ∞ magnifications
- Their images form critical curves

Order of magnitude

For microlensing in the Milky Way, distances ~ few kpc

- **Angular scale:** $\theta_E = \left(\frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}\right)^{1/2} \sim 0.5 \text{ mas}$
- **Einstein radius** $r_E = D_d\theta_E = 2.2 \text{ AU} \left(\frac{M}{0.3\,M_\odot}\right)^{1/2}\left(\frac{D}{2\text{ kpc}}\right)^{1/2}, \; D = \frac{D_d D_{ds}}{D_s}$
- **Timescale:** $t_E = \frac{r_E}{V_t} = 21 \text{ day} \left(\frac{M}{0.3\,M_\odot}\right)^{1/2}\left(\frac{D}{2\text{ kpc}}\right)^{1/2}\left(\frac{V_t}{200\text{ km s}^{-1}}\right)^{-1}$

➔Degeneracy! Can be partially or completely remove for exotic events!

Single lens equation in complex

Normalised lens equation: $\vec{\beta} = \vec{\theta} - \frac{1}{\theta}\frac{\vec{\theta}}{\theta}$

In two dimensions, we write $\vec{\beta} = (x_s, y_s)$, $\vec{\theta} = (x, y)$

$$x_s = x - \frac{x}{x^2 + y^2},$$

$$y_s = y - \frac{y}{x^2 + y^2}, \qquad \times i$$

In complex notation

$$z_s = z - \frac{z}{z\bar{z}} = z - \frac{1}{\bar{z}}, \quad z_s = x_s + y_s\, i, \;\; z = x + y\, i$$

Binary lens equations

Single lens equation:

$$z_s = z - \frac{z}{z\bar{z}} = z - \frac{1}{\bar{z}}, \qquad \text{(m=1, lens at origin)}$$

This can be easily generalised to binary lenses:

$$\mathbf{z}_\mathrm{s} = z - \frac{m_1}{\bar{z} - \bar{z}_1} - \frac{m_2}{\bar{z} - \bar{z}_2}, \quad m_1 + m_2 = 1$$

Taking the conjugate:

$$\bar{\mathbf{z}}_\mathrm{s} = \bar{z} - \frac{m_1}{z - z_1} - \frac{m_2}{z - z_2}.$$

- We obtain zbar and substitute it back into the original equation, which results in a 5th order complex polynomial equation.
- This can be easily solved to find 3 or 5 image positions.

Binary lens magnification

Magnification: $\mu = J^{-1}$

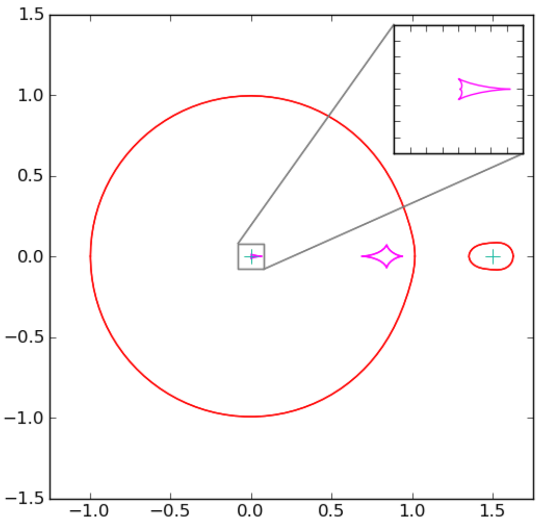
$$J = \det\left(\frac{\partial(x_s, y_s)}{\partial(x, y)}\right) = \det\left(\frac{\partial(z_s, \bar{z}_s)}{\partial(z, \bar{z})}\right)$$

$$= 1 - \left|\frac{m_1}{(z - z_1)^2} + \frac{m_2}{(z - z_2)^2}\right|^2$$

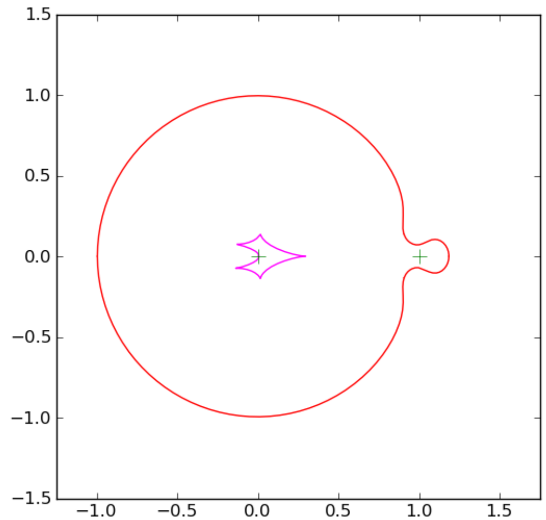
- Critical curves (J=0, $\mu = \infty$) and caustics can then be derived.

Planetary caustics

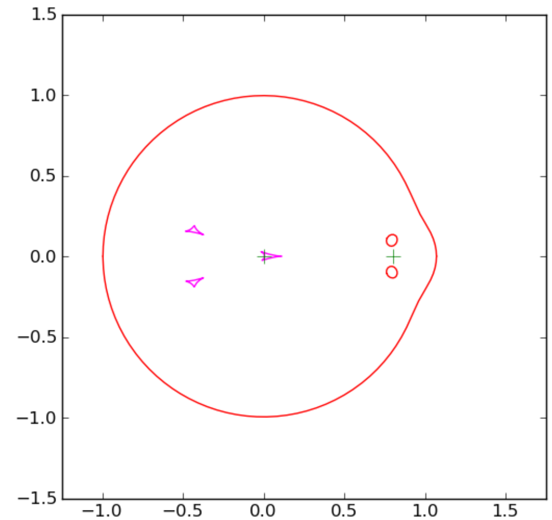
q=0.01, a=1.5



q=0.01, a=1



q=0.01, a=0.8



Far **resonant** **Near**

→

Distance decreases

(Erdl & Schneider 1993)

Principles of binary and exoplanet lensing

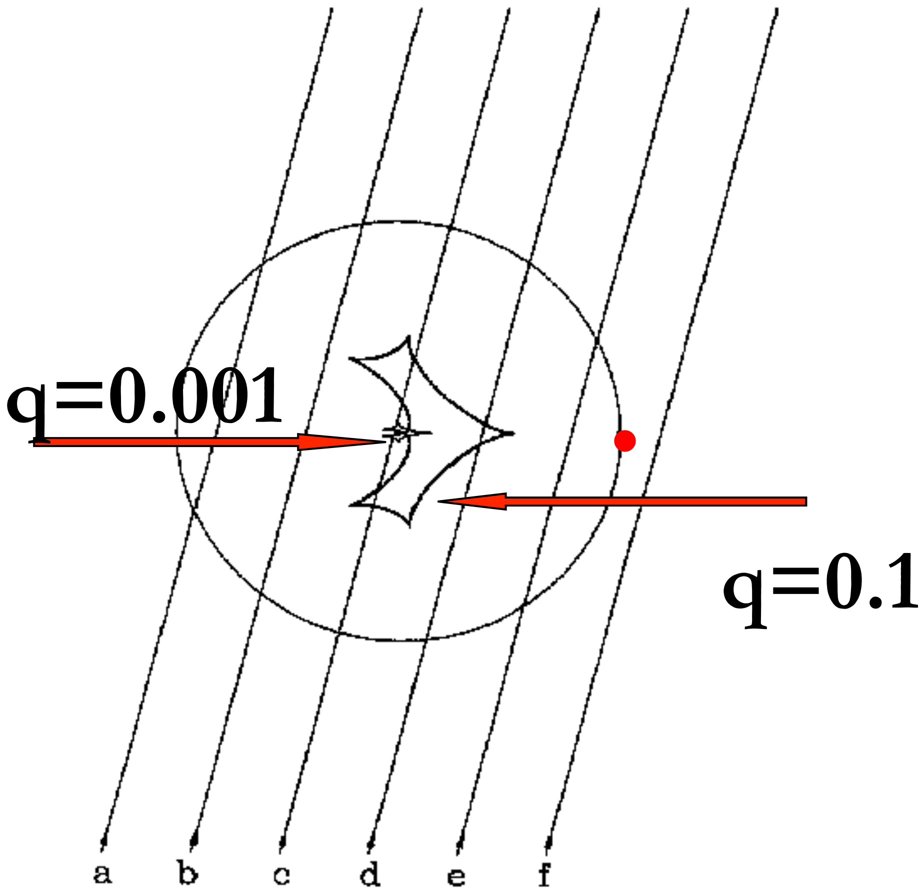
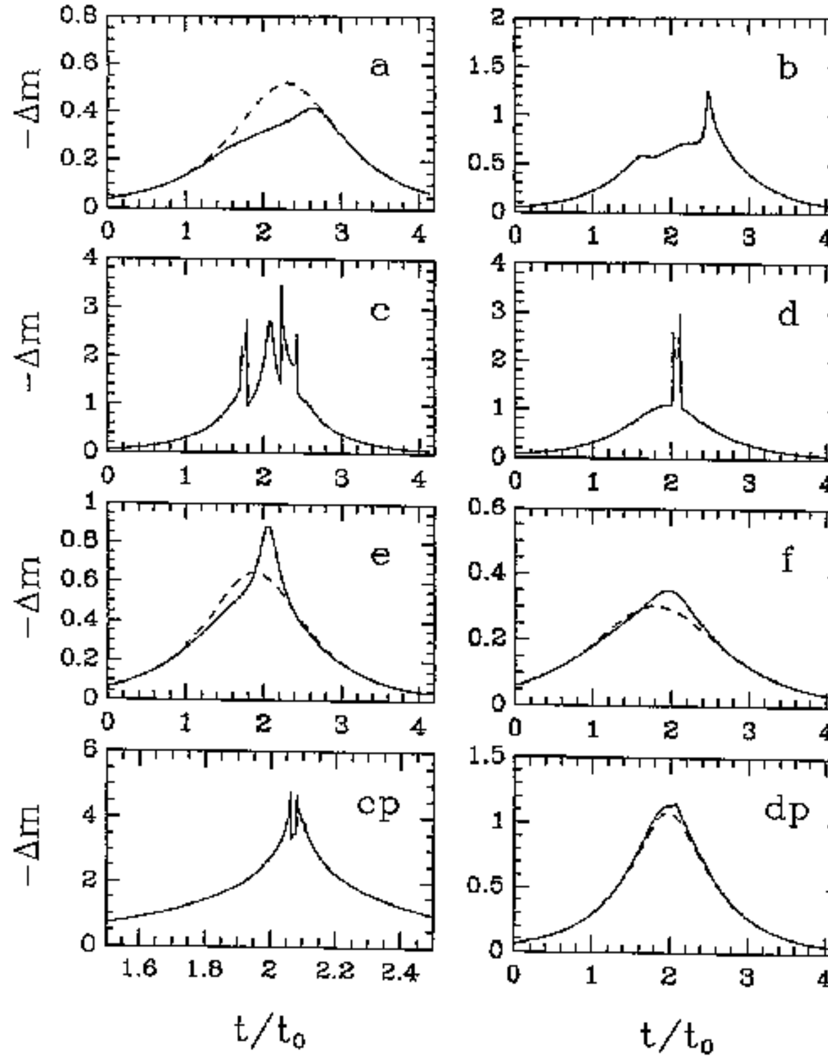


FIG. 1.—Geometry of microlensing by a binary, as seen in the sky. The primary star of 1 $M_\odot$ is located at the center of the figure, and the secondary of 0.1 $M_\odot$ or 0.001 $M_\odot$ is located on the right, on the Einstein ring of the primary. The radius of the ring is 1.0 mas for a source located at a distance of 8 kpc and the lens at 4 kpc. The two complicated shapes around the primary are the caustics; the larger and the smaller corresponding to the 0.1 $M_\odot$ and



(Mao & Paczynski 1991)

Properties of planetary microlensing

- Rate and deviation duration scale roughly ~ (mass ratio)$^{1/2}$
  - Planetary deviation lasts for days for $1M_J$ planets, but hours for $1M_\oplus$ planet
  - Deviation amplitude can be high even for 1 $M_\oplus$ planet
- Microlensing has sensitivity to
  - low-mass planets between ~0.6-1.6 Einstein radii (resonance zone)
  - free-floating planets (seen as single events lasting hours to days)
  - multiple planets
- Complementary to other methods

Summary

- **Gravitational microlensing is "simple", based on GR.**
- **We have derived**
  - **the lens equations, images, and magnifications**
  - **binary lens equation in complex notations**
  - **equations of critical curves and caustics.**
- **Basic principles of extrasolar microlensing**
- **With upgraded/new ground-based (& space) experiments, a particularly exciting decade is ahead!**

Suggested reading

- **General reviews on galactic microlensing:**
  - Mao, S., 2008, "Introduction to microlensing", in Proceedings of the Manchester Microlensing Conference, Edited by E. Kerins, S. Mao, N. Rattenbury, L. Wyzykowski (with exercises!) online at http://pos.sissa.it//archive/conferences/054/002/GMC8_002.pdf
  - This lecture is a simplified version of the ones I gave at a workshop in Italy http://www.jb.man.ac.uk/~smao/lecture1.pdf, http://www.jb.man.ac.uk/~smao/lecture2.pdf, http://www.jb.man.ac.uk/~smao/lecture3.pdf
  - Gould, A., 2008, "Recent Developments in Gravitational Microlensing", presented at "The Variable Universe: A Celebration of Bohdan Paczynski", 29 Sept 2007 online at http://uk.arxiv.org/abs/0803.4324
  - Paczyński, B., 1996, "Gravitational Microlensing in the Local Group", Annual Review of Astronomy and Astrophysics, Vol. 34, p419 online at http://arjournals.annualreviews.org/doi/abs/10.1146/annurev.astro.34.1.419