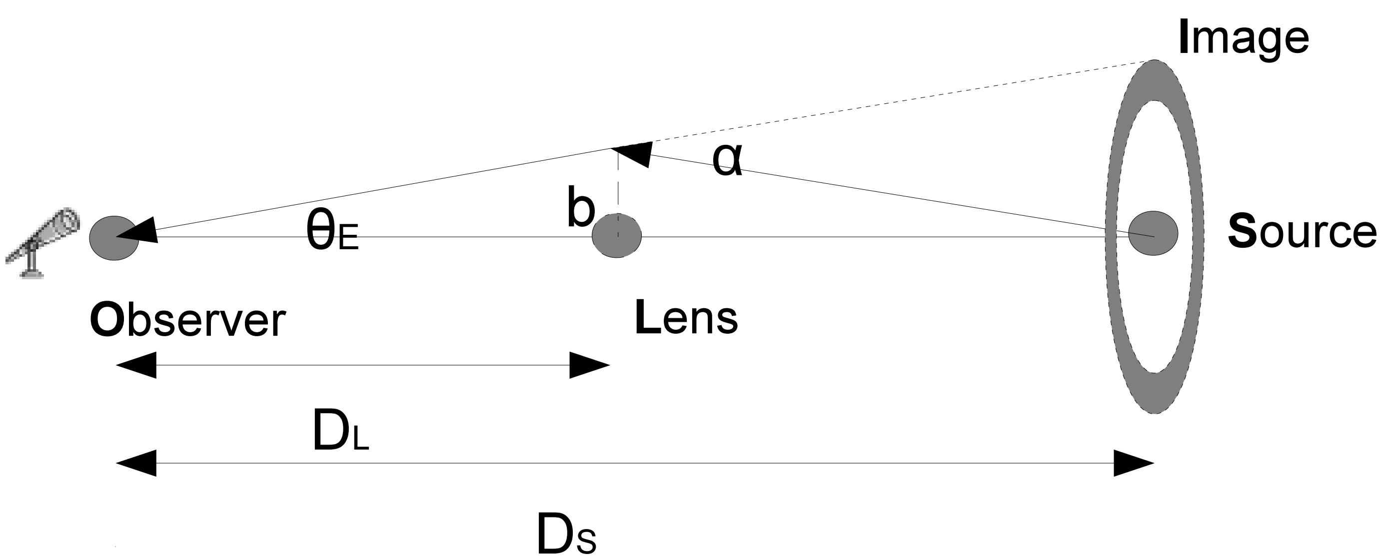
1. If a background star ("source") is lensed by a single star in the foreground ("lens"), what are the image positions θ_I as a function of the source position θ_S?

The only equation you need from GR is the formula for the Einstein bending angle

$$\alpha = \frac{4GM}{bc^2},$$

where M is the mass of the lens and b is the impact parameter.

Let's first consider the simplest case: the source is perfectly aligned with the lens in the line of sight. The image of the source is a ring, called the Einstein ring (see Figure 1). What is the angular size of Einstein ring? Here D_S and D_L are orders of magnitude larger than b, so we can safely use small angle approximation.



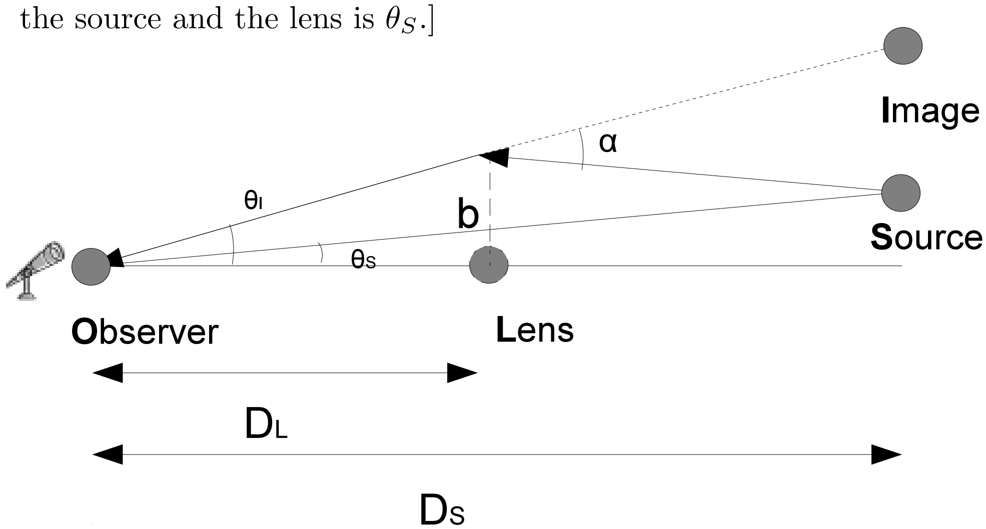
Using simple geometry, we get

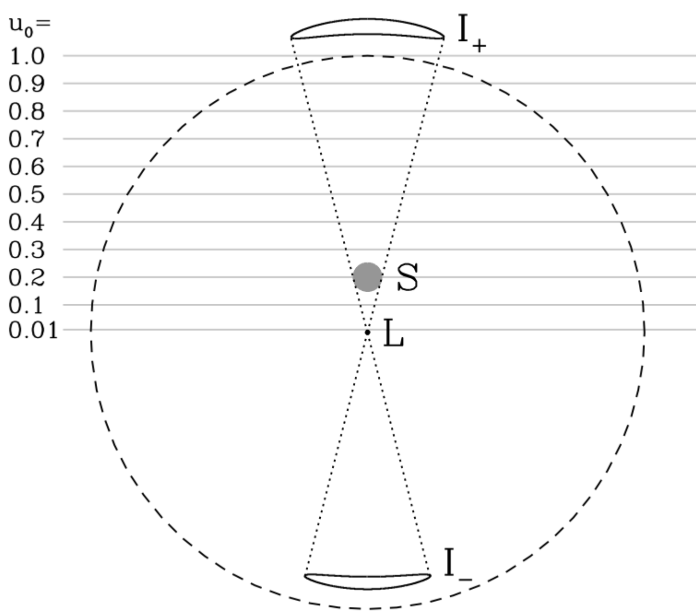
$$\theta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_S - D_L}{D_S D_L}} = \sqrt{\kappa M \pi_{\rm rel}},$$

where constant $\kappa = 4G/c^2\text{AU} = 8.14\,\text{mas}\,M_\odot^{-1}$, M is in $M_\odot$ and relative lens-source parallax $\pi_{\rm rel} = \text{AU}(D_L^{-1} - D_S^{-1})$. **If a background giant star located at the galactic center is lensed by a foreground M-dwarf ($M = 0.5M_\odot$) at 4kpc, what is the**

size of angular Einstein radius? Can the ring be resolved by Keck or TMT? What is the projected physical Einstein radius at the lens distance?

Once the source is misaligned with the lens, as shown in the following figure, we no longer have a ring-like image. Your task is to solve for the image positions. Again, remember to use small angle approximation. [Hint: Show that $\theta_I^2 - \theta_S\theta_I - \theta_{\rm E}^2 = 0$, where θ_I is the angular separation between the images and lens, and the angular distance between the source and the lens is θ_S.]





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From the above diagram, the "physical distance" X from the source to its image is given either by

$$X = \alpha(D_{\rm S} - D_{\rm L})$$

or

$$X = (\theta_I - \theta_S) D_\mathrm{S}$$

Combining these with the formula for Einstein bending angle,

$$\theta_I^2 - \theta_S \theta_I - \theta_\mathrm{E}^2 = 0$$

where

$$\theta_\mathrm{E}^2 \equiv \frac{4GM(D_\mathrm{S} - D_\mathrm{L})}{D_\mathrm{L} D_\mathrm{S} c^2}$$

is the Einstein radius we derived earlier. There are two solutions, so two images at

$$\theta_{I,\pm} = y_\pm \theta_\mathrm{E}; \qquad y_\pm = \frac{u \pm \sqrt{u^2+4}}{2}; \qquad u \equiv \frac{\theta_S}{\theta_\mathrm{E}}$$

Surface brightness is conserved in lensing, so the magnification equals the angular area of the image divided by that of the source. The magnifications for the two images are:

$$A_\pm = \left| \frac{y_\pm}{u} \frac{\mathrm{dy}_\pm}{\mathrm{du}} \right| = \frac{1}{2} \left[\frac{u^2+2}{u\sqrt{u^2+4}} \pm 1 \right]$$

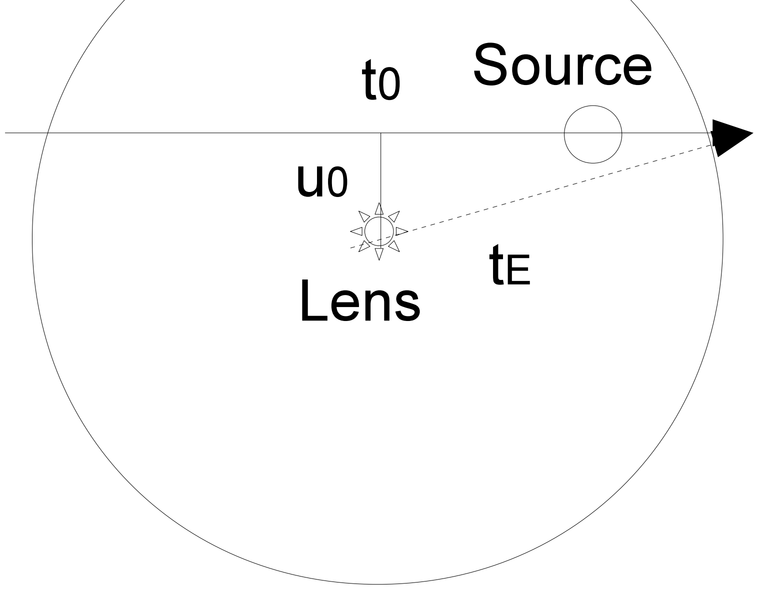
.

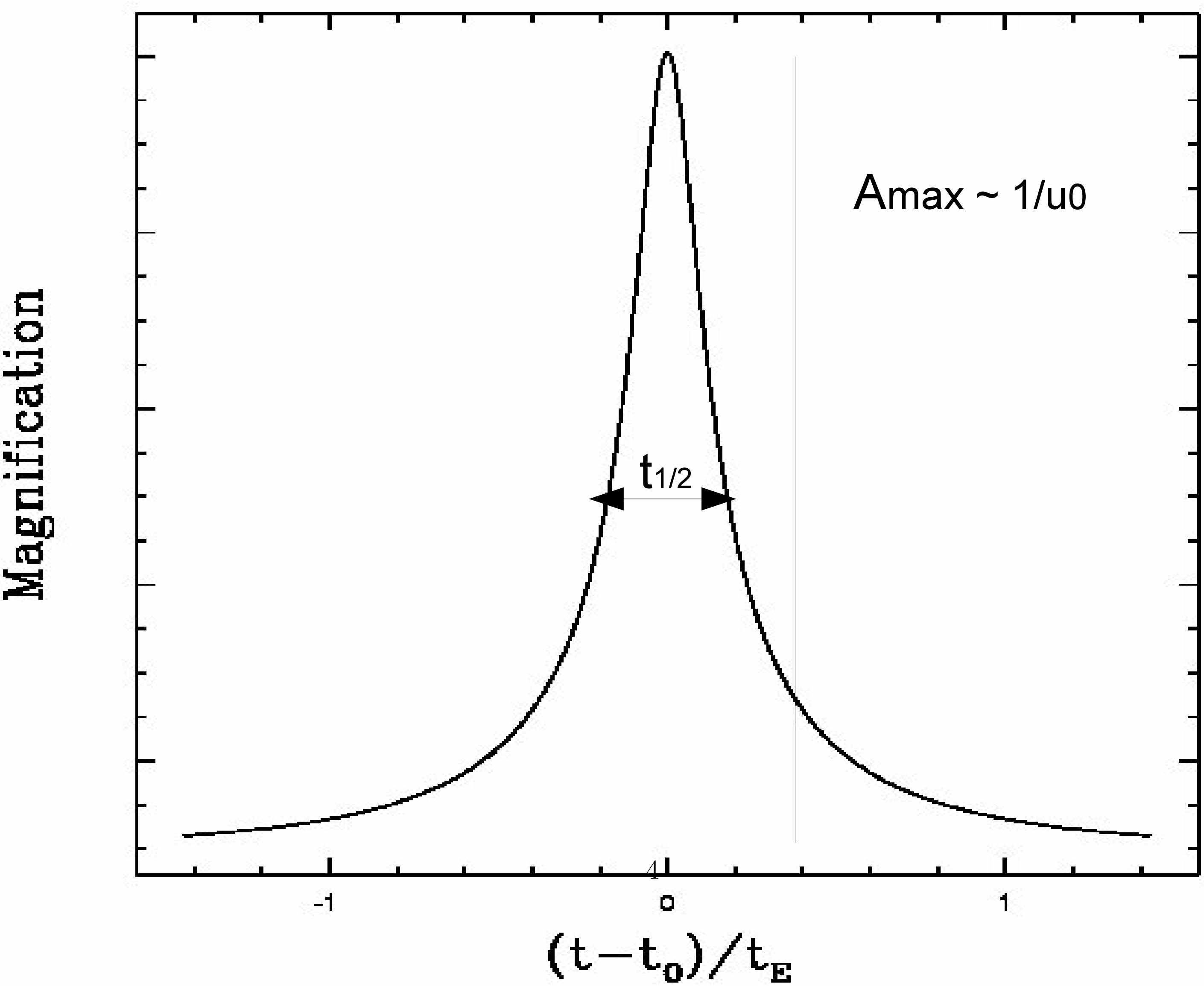
Thus the total magnification is

$$A = \frac{u^2+2}{u\sqrt{u^2+4}}$$

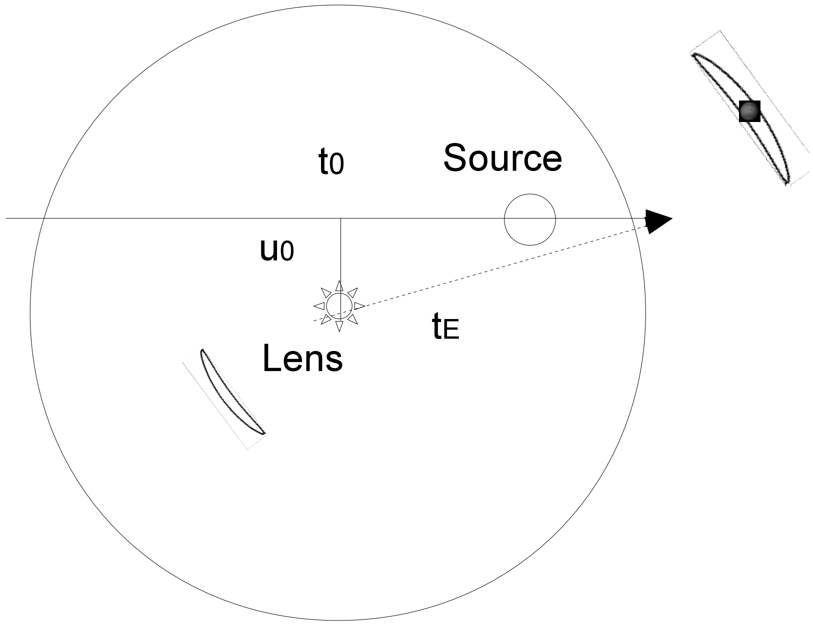
When u is small, $A \sim 1/u$. This is good enough for today's eyeball modeling.

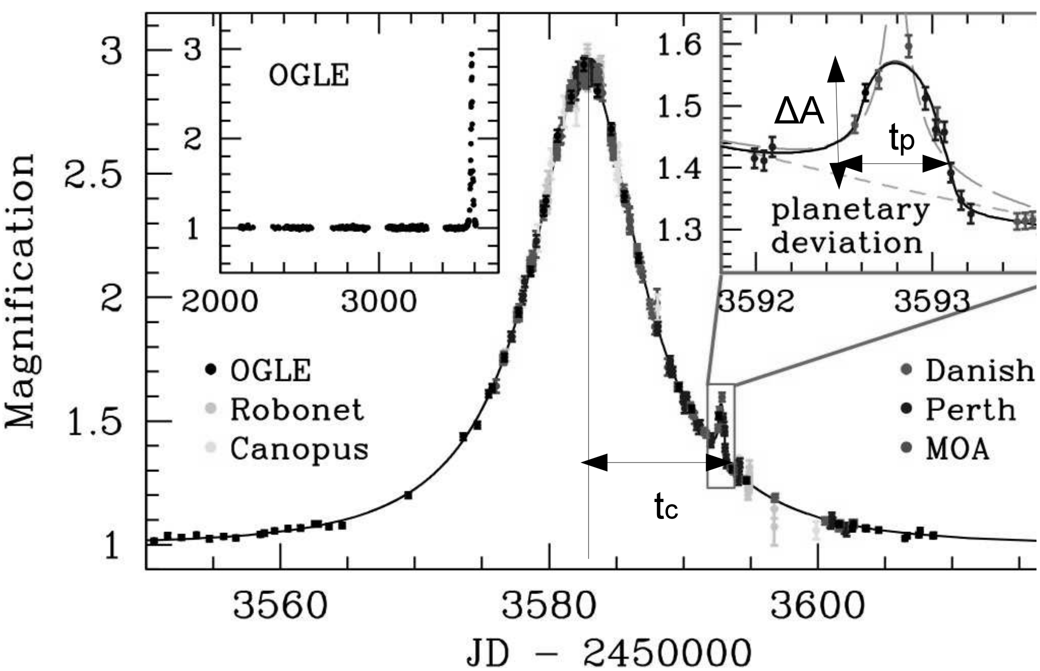
The following figure shows the relevant geometry and the corresponding light curve. The light curve can be described by three parameters: time of the peak magnification t_0, impact parameter u_0 and Einstein radius crossing time t_E. If u_0 is small, how would you quickly estimate t_E just by looking at the light curve? [Hint: Calulate the relation between $t_{1/2}$ and t_E, where $t_{1/2}$ is the time span at half peak magnification.]


Source
t0
u0
tE
Lens


Amax ~ 1/u0
t1/2
Magnification
-1
0
1
(t−to)/tE

2. The next figure shows the light curve of OGLE-2005-BLG-390Lb, a planet discovered by PLANET and OGLE collaborations in 2005. The planetary signal occurs when the planet sits right on top of the (major) image outside the Einstein ring. What is the projected distance between the planet and the star in the units of Einstein radius?





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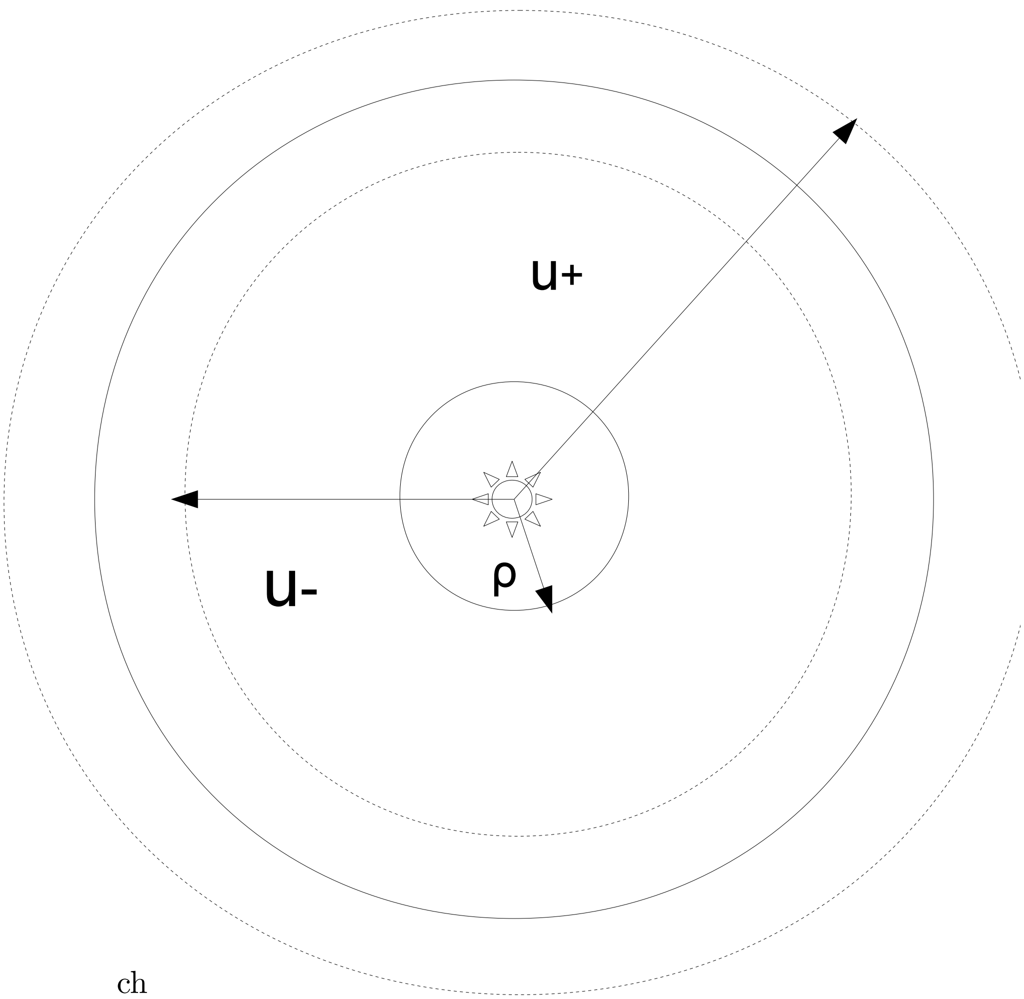
3. What is the magnification of a uniform source of angular radius $\theta_* = \rho\theta_E$, sitting right on top of the lens? What is the "excess magnification" due to a lens in the limit that the source is much bigger than the Einstein ring? [Hint: Surface brightness is conserved in lensing, so the magnification equals the angular area of the image divided by that of the source.]



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The source boundary will be imaged into two rings at $y_\pm = (u \pm \sqrt{u^2+4})/2$. So the area of the image is $\pi(y_+^2 - y_-^2)$. Explicitly,

$$A = \frac{\pi(y_+^2 - y_-^2)}{\pi\rho^2} = \sqrt{1+\frac{4}{\rho^2}}$$

Clearly, if $\rho >> 1$, $A \to 1 + 2/\rho^2$, so the excess magnification is just $2/\rho^2$.

4. In this event, the angular size of the source is much smaller than the angular Einstein radius of the lens star, but much larger than that of the planet. From the light curve features, derive planet to star mass ratio q.

Step 1. The planet Einstein radius is $\theta_p = \sqrt{4GmD_{\rm LS}/D_{\rm L}D_{\rm S}c^2}$. Hence, the mass ratio q is

$$q \equiv \frac{m}{M} = \frac{\theta_p^2}{\theta_{\rm E}^2}$$

A good conjecture, which turns out to be correct, is that the excess magnification due to the planet is

$$\Delta A_p = 2\left(\frac{\theta_p}{\theta_*}\right)^2$$

Step 2. What is the relationship between the duration of the planetary perturbation t_p, the Einstein crossing time $t_{\rm E}$, the source size θ_* and the Einstein radius $\theta_{\rm E}$?

Clearly $\theta_{\rm E}/t_{\rm E}$ and $2\,\theta_*/t_p$ are both equal to the proper motion, so they are equal to each other:

$$\frac{t_p}{t_{\rm E}} = \frac{2\theta_*}{\theta_{\rm E}}$$

Step 3. Estimate q in terms of observables.

$$q \equiv \frac{m}{M} = \frac{\theta_p^2}{\theta_{\rm E}^2} = \frac{\theta_p^2}{\theta_*^2}\frac{\theta_*^2}{\theta_{\rm E}^2} = \frac{\Delta A_p}{2}\,\frac{t_p^2}{(2t_{\rm E})^2}$$

Step 4. What would be the excess magnification caused by an Earth-mass planet $q = 10^{-5}$? What would that be if the angular source size is ten times smaller?

References:

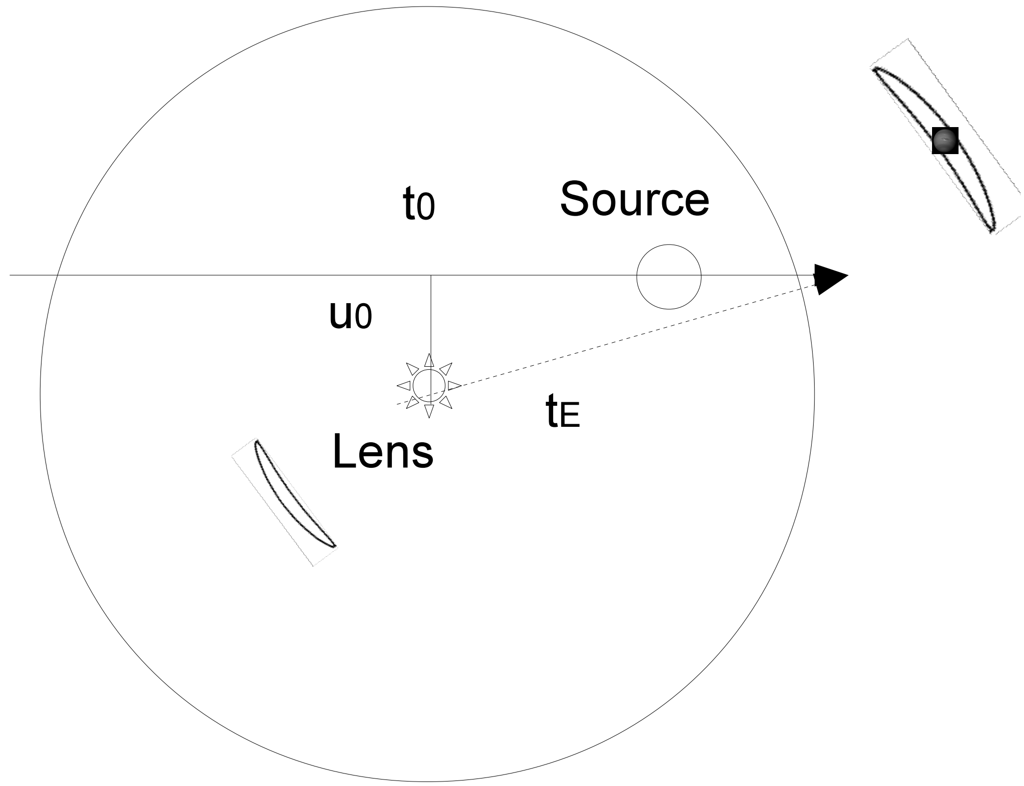
1. A good general review of microlensing and basic maths that goes into it.
Shude Mao: Introduction to Gravitational Microlensing
http://www.arxiv.org/abs/0811.0441

2. A comprehensive review article on planetary microlensing.
B. Scott Gaudi: Exoplanetary Microlensing

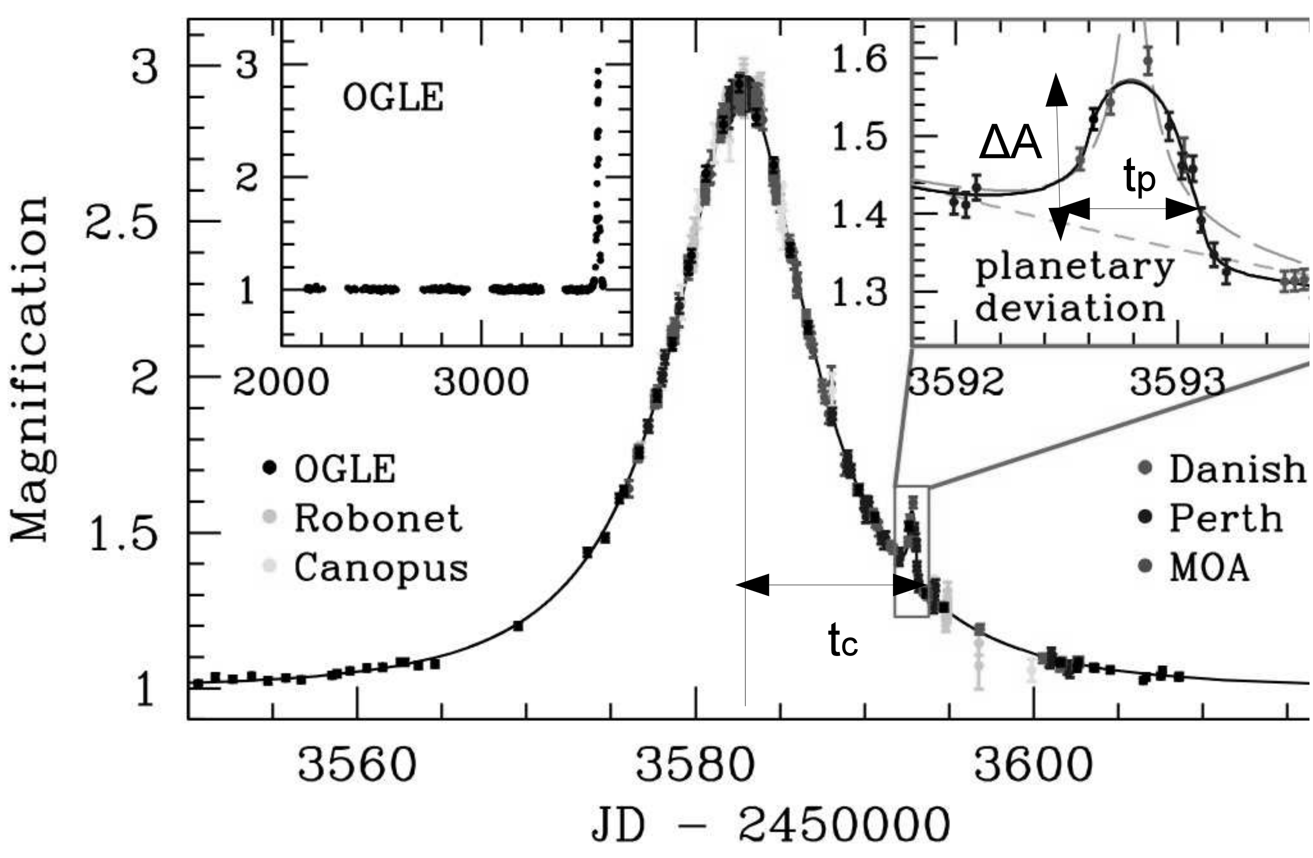
http://www.arxiv.org/abs/1002.0332

3. Answer to your homework problem.
Alison Farmer, Eric Agol: Finding White Dwarfs with Transit Searches
http://www.arxiv.org/abs/astro-ph/0303453

4. The discovery paper of OGLE-2005-BLG-390Lb.
Beaulieu et al., 2006, Nature, 439, 440.
http://www.arxiv.org/abs/astro-ph/0601563



Source
t_0
u_0
t_E
Lens



Magnification
JD − 2450000
OGLE
OGLE
Robonet
Canopus
Danish
Perth
MOA
ΔA
t_p
planetary deviation
t_c