Shaped Pupil Coronagraphs

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Electric Field (Fourier Optics)

The image-plane *electric field* $E()$ produced by an on-axis plane wave and an apodized aperture defined by an *apodization function* $A()$ is given by

$$
\begin{aligned}
E(\xi,\zeta) &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-2\pi i(x\xi+y\zeta)} A(x,y)\,dy\,dx \\
&\vdots \\
E(\rho) &= 2\pi \int_0^{1/2} J_0(2\pi r\rho) A(r) r\,dr,
\end{aligned}
$$

where J_0 denotes the 0-th order Bessel function of the first kind.

The unitless pupil-plane "length" r is given as a multiple of the aperture D.

The unitless image-plane "length" ρ is given as a multiple of focal-length times wavelength over aperture $(f\lambda/D)$ or, equivalently, as an angular measure on the sky, in which case it is a multiple of just λ/D.

The *point spread function* (psf) is the square of the electric field.

Performance Metrics

Inner and Outer Working Angles

$$\rho_{\text{iwa}} \qquad \rho_{\text{owa}}$$

Contrast:

$$E^2(\rho)/E^2(0)$$

Integration Time / Throughput

Some Throughput Measures

Throughput is a surrogate for integration time

Airy Throughput

$$\mathcal{T}_{Airy} = \frac{\int_0^{\rho_{\mathsf{iwa}}} E^2(\rho) 2\pi \rho d\rho}{(\pi(1/2)^2)} = 8 \int_0^{\rho_{\mathsf{iwa}}} E^2(\rho)\rho d\rho.$$

or,

$$\mathcal{T}_{Airy} = 4 \int_0^{\xi_{iwa}} \int_0^{\zeta_{iwa}} E^2(\xi, \zeta) d\zeta d\xi$$

Total Throughput

$$\mathcal{T}_{total} = 4 \int_0^{1/2} \int_0^{1/2} A^2(x, y) dx dy$$

Central Throughput

$$\mathcal{T}_{central} = E(0) = 2\pi \int_0^{1/2} A(r) dr$$

Some Integration Time Formulas

Reference Integration Time (Perfect System / No Background)

$$t_0 = \frac{(S/N)^2}{I_p A_0}$$

Known Background

$$\frac{t_1}{t_0} = \frac{A_0(P(0,0)\Delta S + Q\sum \Delta P_{ij})}{Q(\sum \Delta P_{ij})^2} = \frac{1}{\mathcal{T}_{\mathsf{total}}} \frac{\Delta S + Q\sum \Delta \bar{P}_{ij}}{Q(\sum \Delta \bar{P}_{ij})^2}$$

Photometric Estimation (Known Background)

$$\frac{t_2}{t_0} = \frac{1}{\mathcal{T}_{\mathsf{total}}} \frac{\Delta\alpha \sum \Delta \bar{P}_{ij}^2 + Q\sum \Delta \bar{P}_{ij}^3}{Q(\sum \Delta \bar{P}_{ij}^2)^2}$$

Photometric Estimation (Unknown Background)

$$\frac{t_3}{t_0} = \frac{1}{\mathcal{T}_{\mathsf{total}}} \frac{Q\Delta\alpha^2[\Delta S^2 \sum \Delta P_{ij}^3 - \Delta\alpha^2(\sum \Delta P_{ij})^3] + \Delta S \Delta\alpha^2/P^2(0,0)}{Q[\Delta\alpha \Delta S \sum \Delta P_{ij}^2 - \Delta\alpha^2(\sum \Delta P_{ij})^2]^2}$$

The Optimization Problem

In general, it is too difficult to optimize integration time (or sharpness) directly. Instead, we optimize its surrogate, throughput.

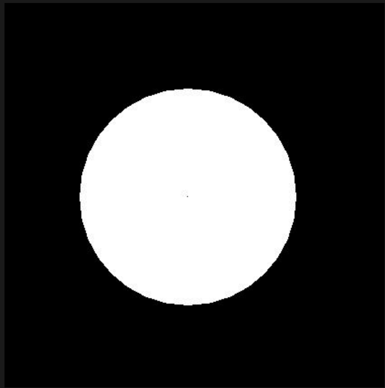
Thus, an optimal apodization problem is to find the function $A(x, y)$ (or $A(r)$) that maximizes a measure of throughput subject to constraints on contrast at the desired inner and outer working angles.

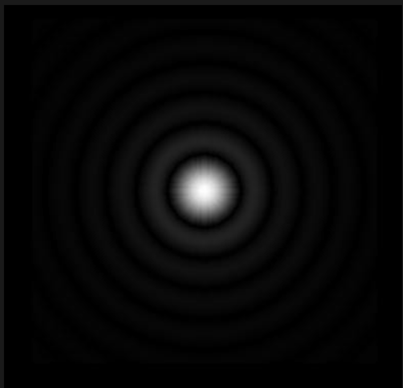
First, some optimal smooth apodizations . . .

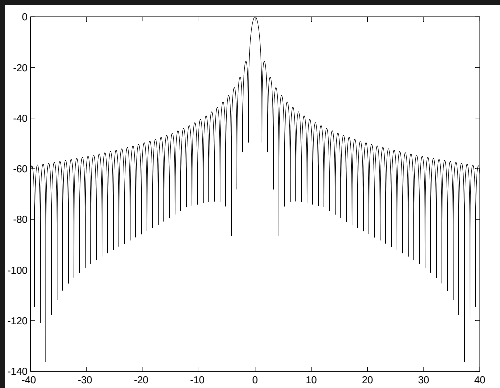
Clear Aperture

$$\mathrm{FWHM} = 1.02 \quad \rho_{\mathsf{iwa}} = 1.24 \quad \mathcal{T}_{\mathsf{Airy}} = 84.2\%$$

No dark zone.







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Infinite 1-D Optimization Problem

For an infinite aperture, the minimum variance aperture is a Gaussian by the uncertainty principal:

$$(\Delta x)^2 = \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \Big/ \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$(\Delta s)^2 = \int_{-\infty}^{\infty} s^2 |F(s)|^2 ds \Big/ \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Uncertainty Principal: $(\Delta x)(\Delta s) \geq \frac{1}{4\pi}$

Equality is achieved for Gaussian apodization.

However, truncating the Gaussian for finite apertures introduces significant side-lobes.

The Finite 1-D Optimization Problem

$$\text{maximize} \quad \frac{\int_{-\Omega}^{\Omega} E(\xi,0)^2 d\xi}{\int_{-\infty}^{\infty} E(\xi,0)^2 d\xi}$$

or, equivalently,

$$\text{minimize} \quad \int_{\Omega}^{\infty} E(\xi,0)^2 d\xi$$
$$\text{subject to} \quad A(0) = 1.$$

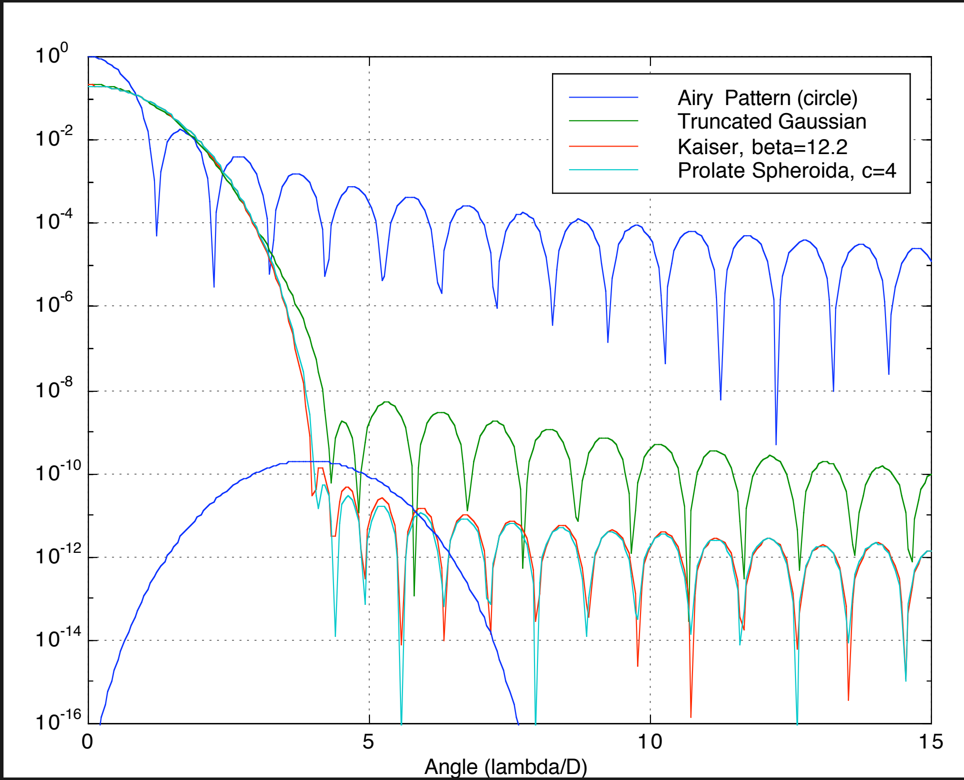
Slepian (1961) solved this by formulating a finite uncertainty principal, equality of which is found by solving the following eigenvalue equation:

$$\lambda f(t) = \frac{1}{\pi} \int_{-T/2}^{T/2} f(s) \frac{\sin \Omega (t-s)}{(t-s)} ds$$

Solution is the Prolate Spheroidal Wavefunction.

This can also be solved via the calculus of variations.

The Prolate Spheroidal Wavefunction



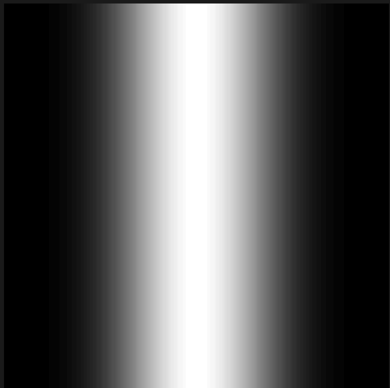
For a contrast of 10^{-10}, the inner working angle is 4 λ/D.

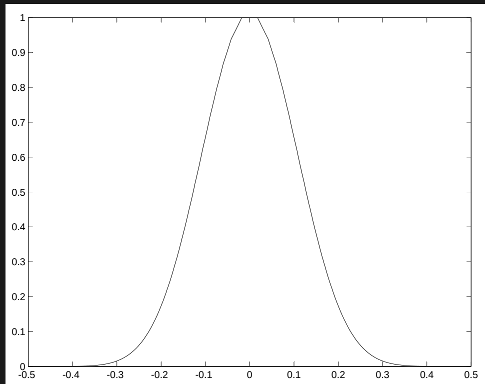
1-D Apodization (Square)

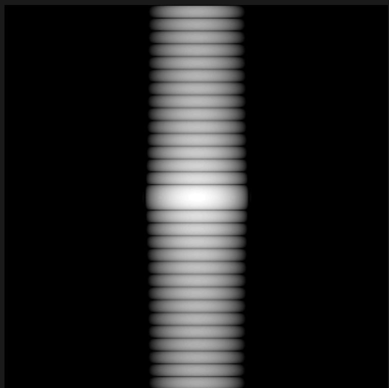
Slepian's Prolate Spheroidal Wavefunction ($\psi_0(x)$) (1965)

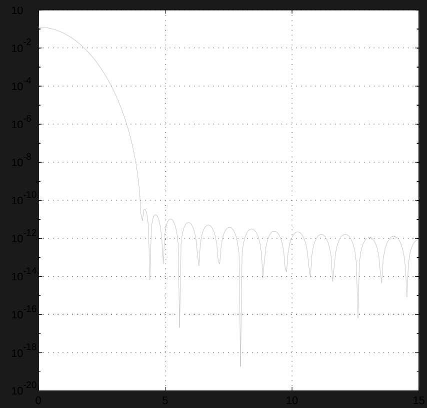
$$\text{FWHM} = 2 \quad \rho_{\text{iwa}} = 4 \quad \mathcal{T}_{\text{Airy}} = 25\%$$

Good dark zone. Unmanufacturable.









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A Direct Optimization Problem

We can formulate a related max-pseudo-area problem:

$$
\begin{array}{lrcll}
\text{maximize} & \int_0^{1/2} A(x)dx & & & \\
\text{subject to} & -10^{-5}E(0,\zeta) \leq & E(0,\zeta) & \leq 10^{-5}E(0,\zeta), & \xi_{\text{iwa}} \leq \xi \leq \xi \\
 & 0 \leq & A(x) & \leq 1, & 0 \leq x \leq 1/ \\
 & & A'(x) & \leq 0, & 0 \leq x \leq 1/2 \\
 & & A(x)A''(x) & \leq A'(x)^2, & 0 \leq x \leq 1/2
\end{array}
$$

This problem can be discretized to a linear program and very efficiently solved numerically.

Solution is very similar to the prolate spheroidal apodization.

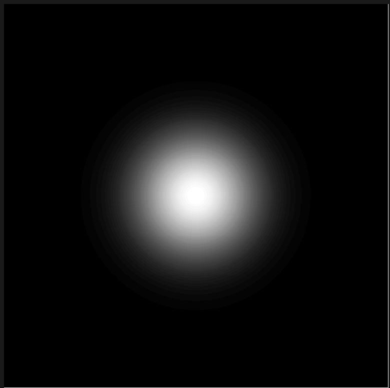
We can formulate a very similar problem in polar coordinates . . .

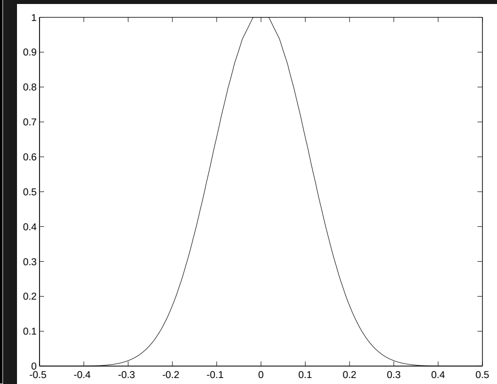
2-D Apodization (Circle)

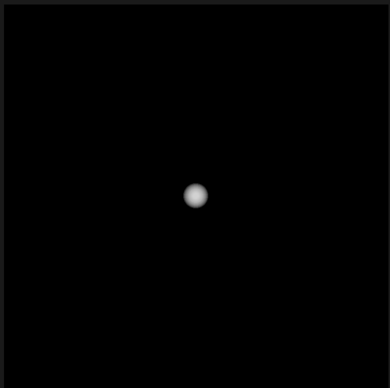
Slepian's Generalized Prolate Spheroidal Wavefunction ($\psi_0(x)$)

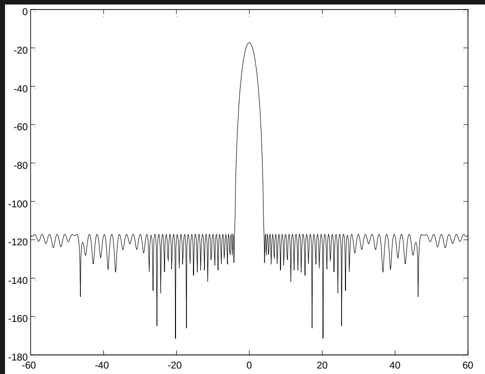
$$\text{FWHM} = 2 \quad \rho_{\text{iwa}} = 4 \quad \mathcal{T}_{\text{Airy}} = 9\%$$

Excellent dark zone. Unmanufacturable.









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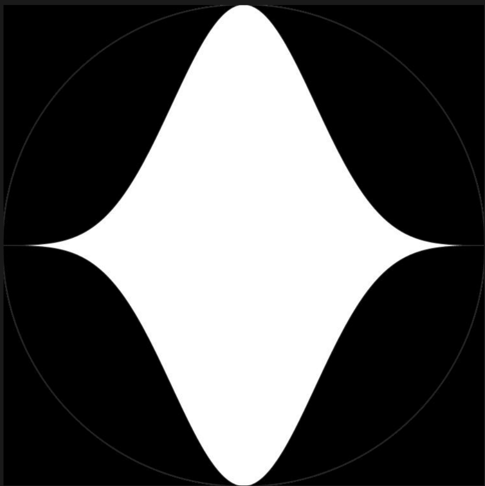
Single Shaped Pupil

On axis Electric Field is the same 1-D Fourier Transform:

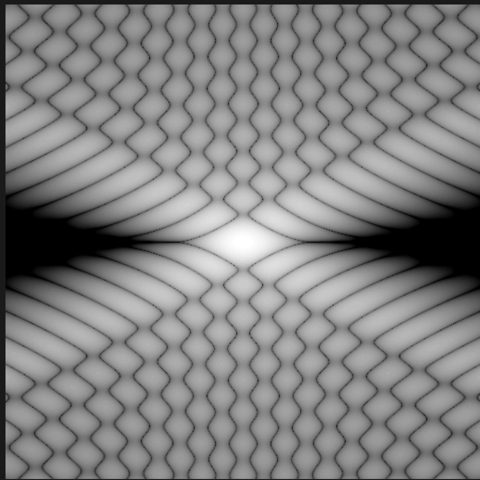
$$E(\xi, 0) = \int_{-1/2}^{1/2} w(x) e^{-i2\pi\xi x} dx$$

$$\mathrm{FWHM} = 1.9 \quad \rho_{\mathrm{iwa}} = 4 \quad \mathcal{T}_{\mathrm{Airy}} = 43\%$$

Small dark zone...Many rotations required



PSF for Single Prolate Spheroidal Pupil



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Larger Discovery Zone

We can open up the discovery space by stacking many openings.
Can reproduce any apodization.
For N openings, the electric field is:

$$E(\xi,\zeta) = \frac{2\sin(\pi\zeta)}{\pi\zeta}\int_0^{1/2}\left(\begin{array}{c}\cos(\pi\zeta w(x)) \\ -\frac{[1+\cos(\pi\zeta/N)]}{\sin(\pi\zeta/N)}\sin(\pi\zeta w(x))\end{array}\right)\cos(2\pi\xi x)dx$$

Expanding about $\zeta = 0$, we identify:

$$w(x) = \frac{1-A(x)}{2N}$$

which for large N results in,

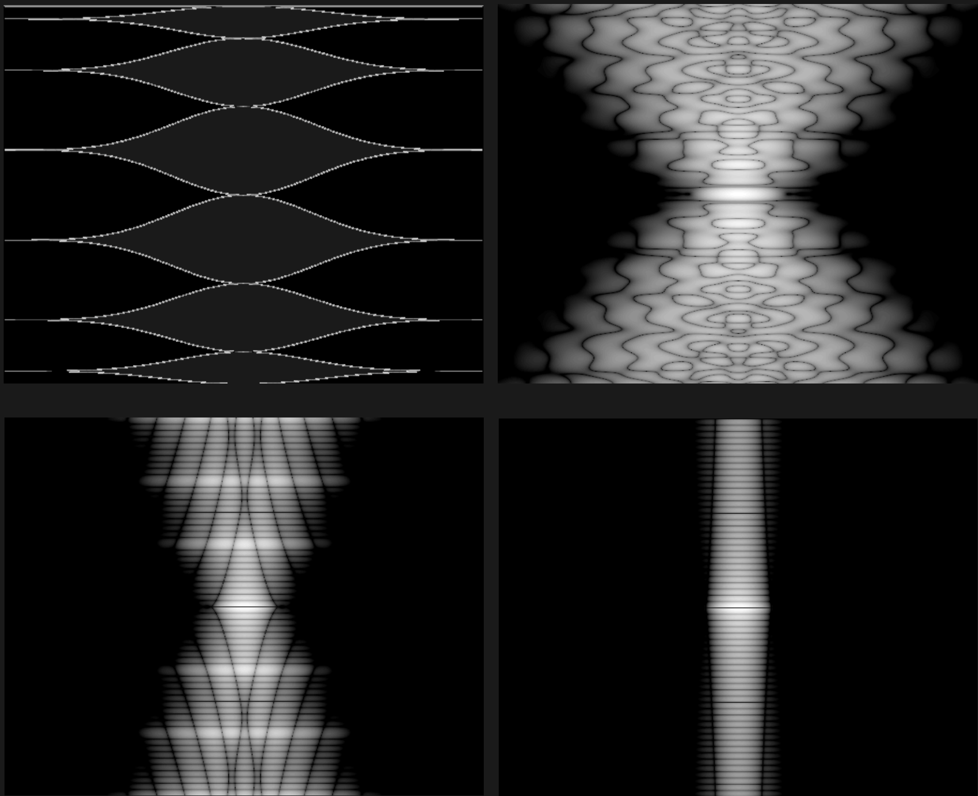
$$E(\xi,\zeta) = \frac{2\sin(\pi\zeta)}{\pi\zeta}\int_0^{1/2}\left(A(x) + \mathcal{O}\left(\frac{\pi\zeta}{N}\right)^2\right)\cos(2\pi\xi x)dx$$

For N large enough, diffracted light from openings can be made arbitrarily small within discovery region.

Multiple Pupil Mask (repeated)

$$\text{FWHM} = 1.9 \quad \rho_{\text{iwa}} = 4$$

Good dark zone. Possible to Make. Any Apodization.



10 Pupil Mask 100 Pupil Mask

Starshaped Masks

The polar equivalent multi-pupil mask.

Electric field here is given by:

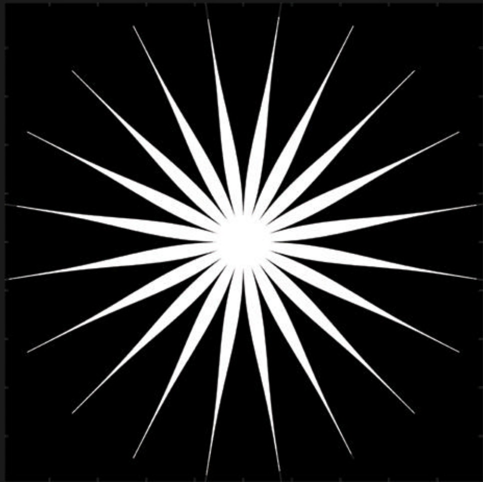
$$E(\rho,\phi) = 2\pi \int_0^{1/2} J_0(2\pi r\rho)(1 - \frac{N}{2\pi}\alpha(r))rdr$$
$$-4\sum_{j=1}^{\infty} \int_0^{1/2} J_{jN}(2\pi r\rho)\cos(jN(\phi - \pi/2))\frac{1}{j}\sin(jN\frac{\alpha(r)}{2})rdr$$

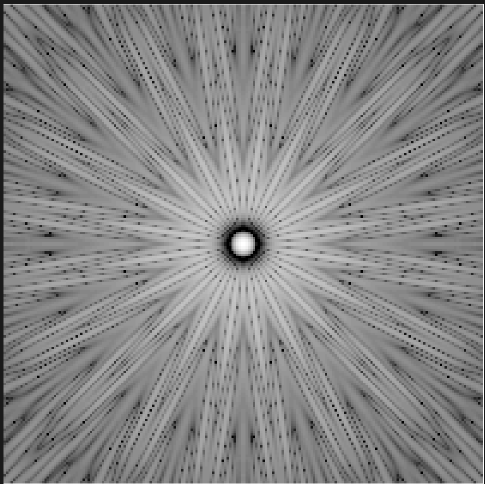
Second term throws light outside the owa for large enough N.

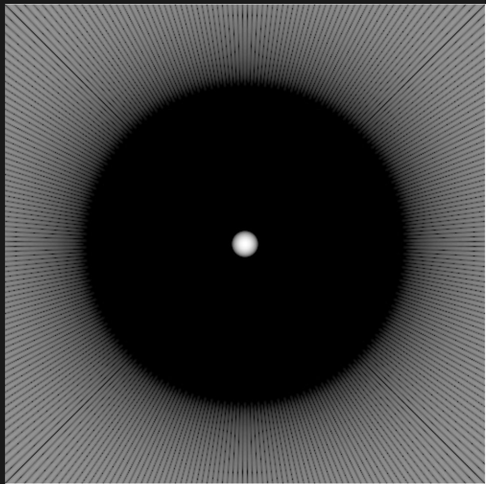
Starshaped Masks: 20 and 150 Points

Any apodization works:

$$\alpha(r) = \frac{2\pi}{N} A(r)$$

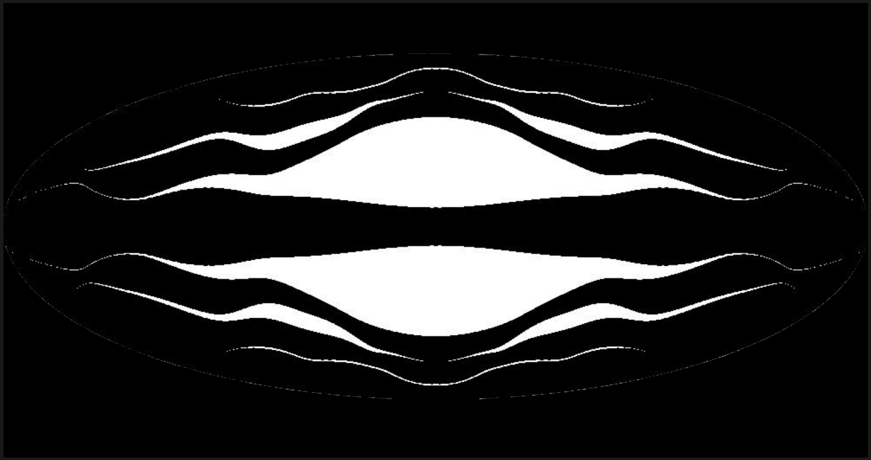


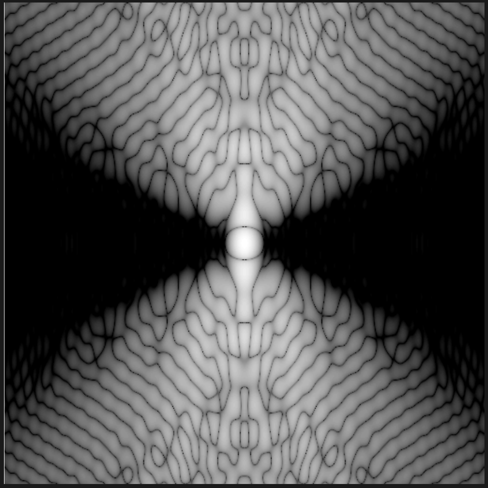




Multiple Pupil Mask (direct opt.)

We can also directly optimize the shapes of each opening.





$\mathrm{FWHM} = 2.0 \quad \rho_{\text{iwa}} = 4$

$\mathcal{T}_{\text{Airy}} = 30\%$

Throughput relative to ellipse

11% central obstr.

Easy to make

Very few rotations

Binary Apodizations

What if we remove smoothness constraints?

$$\begin{array}{ll} \text{maximize} & \int_0^{1/2} A(x)dx \\ \text{subject to} & -10^{-5}E(0,\zeta) \leq E(0,\zeta) \leq 10^{-5}E(0,\zeta), \quad \xi_{\text{iwa}} \leq \xi \leq \xi_{\text{owa}}, \\ & \qquad 0 \leq \; A(x) \; \leq 1, \qquad\qquad 0 \leq x \leq 1/2, \end{array}$$

It turns out that the numerical solution of this problem is zero-one valued!

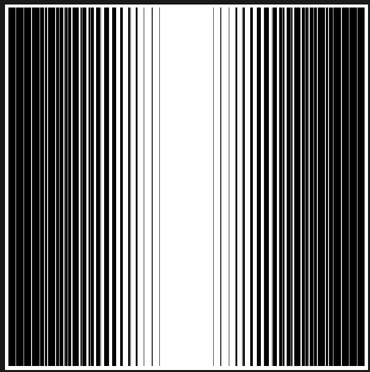
The optimal apodization is binary, i.e., a shaped pupil.

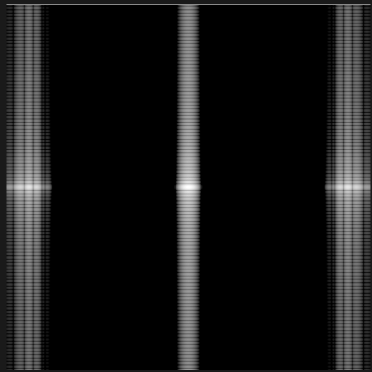
In 1-D, we call the resulting mask a *barcode mask*.

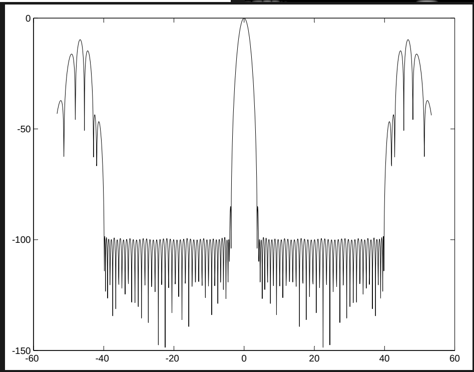
Barcode Mask

$$\text{FWHM} = 2.1 \quad \rho_{\mathsf{iwa}} = 4 \quad \rho_{\mathsf{owa}} = 40$$

$$\mathcal{T}_{\mathsf{Airy}} = 25\% \quad \mathsf{Contrast} = 10^{-10}$$







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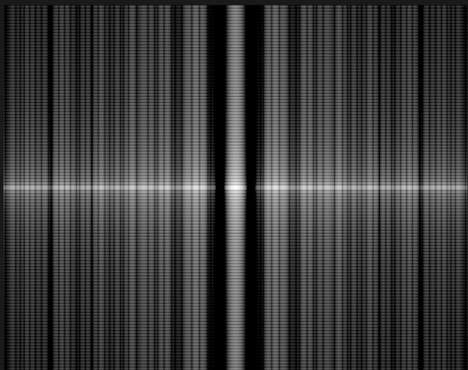
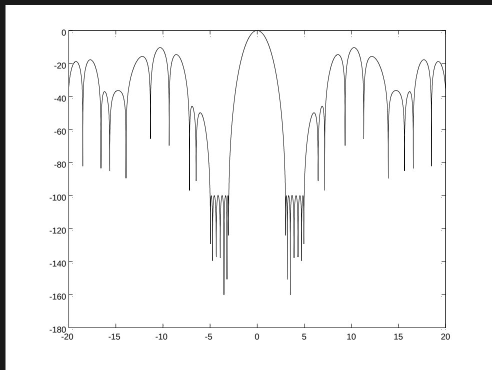
Barcode Mask w/ Smaller iwa

$$\mathrm{FWHM} = 2.1 \quad \rho_{\mathsf{iwa}} = 3 \quad \rho_{\mathsf{owa}} = 5$$

$$\mathcal{T}_{\mathsf{Airy}} = 33.8\% \quad \mathsf{Contrast} = 10^{-10}$$





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Concentric Ring Mask

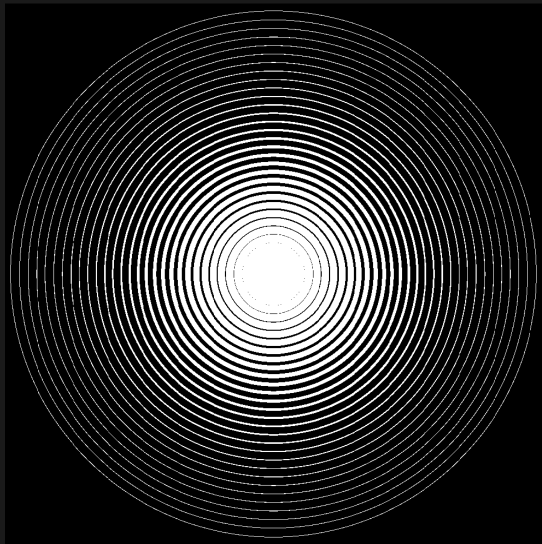
$$\text{FWHM} = 1.9$$

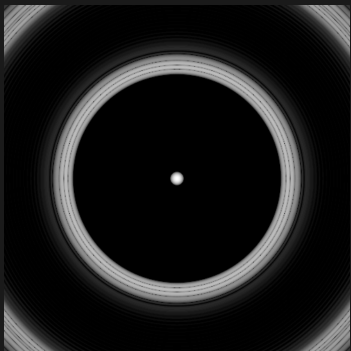
$$\rho_{\text{iwa}} = 4 \quad \rho_{\text{owa}} = 60$$

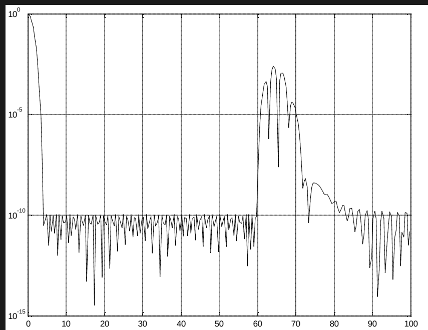
$$\mathcal{T}_{\text{Airy}} = 9\%$$

Lay it on glass?

No rotations







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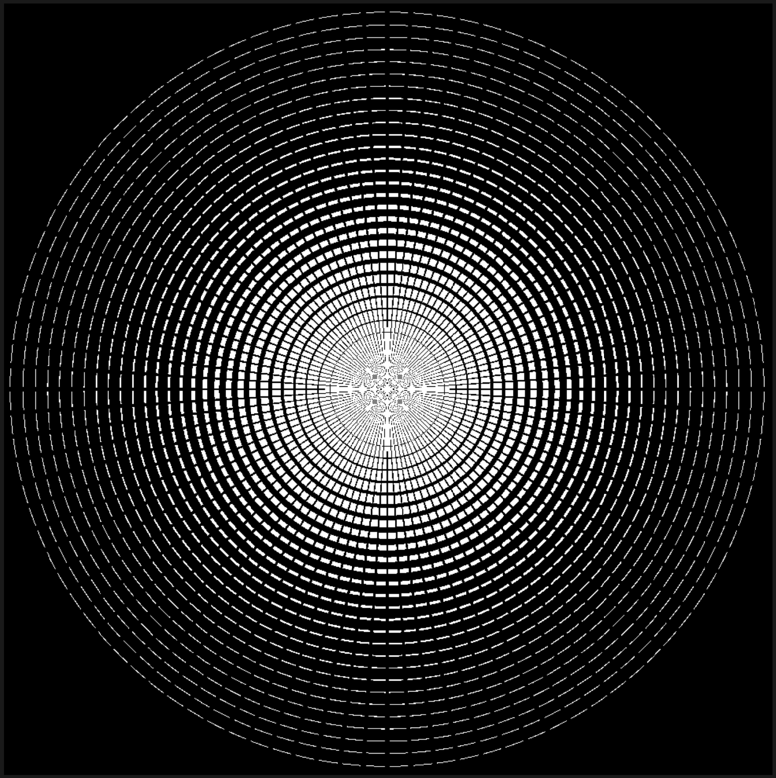
Spiderweb Mask: 100 Vanes

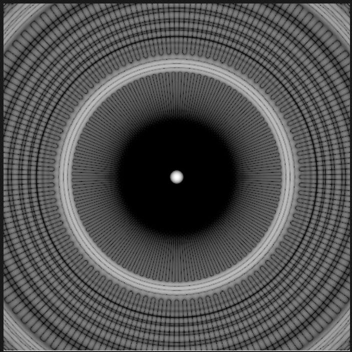
$$\text{FWHM} = 1.9 \quad \rho_{\text{iwa}} = 4 \quad \rho_{\text{owa}} = 40 \quad \mathcal{T}_{\text{Airy}} = 9(1-\theta)\%$$

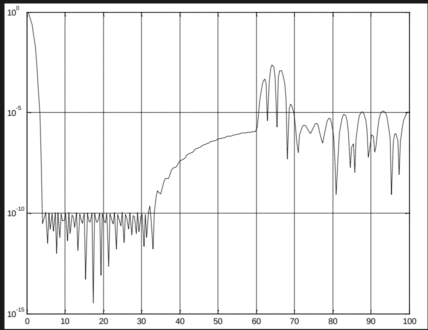
Manufacturable?

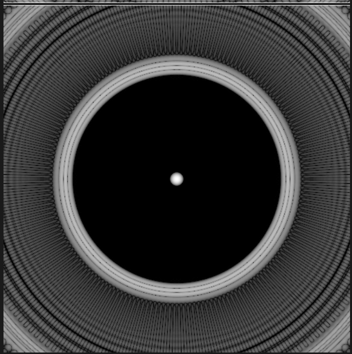


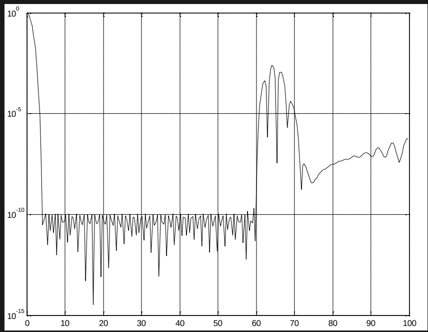
Spiderweb Mask: 100 and 180 Vanes

$$\text{FWHM} = 2 \quad \rho_{\text{iwa}} = 4 \quad \rho_{\text{owa}} = 40 \quad \mathcal{T}_{\text{Airy}} = 9(1-\theta)\%$$







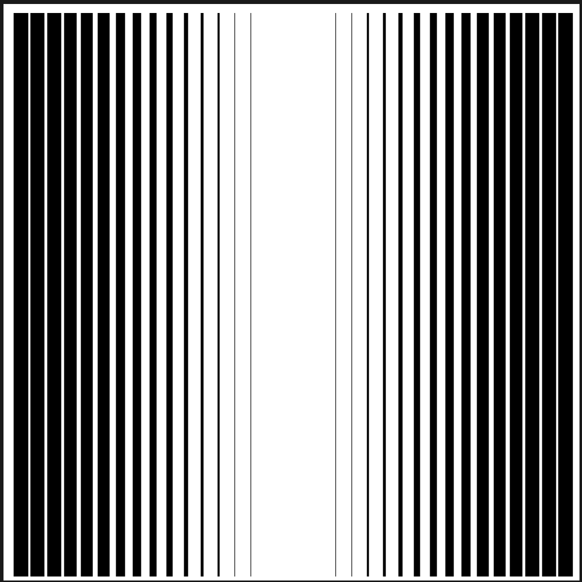


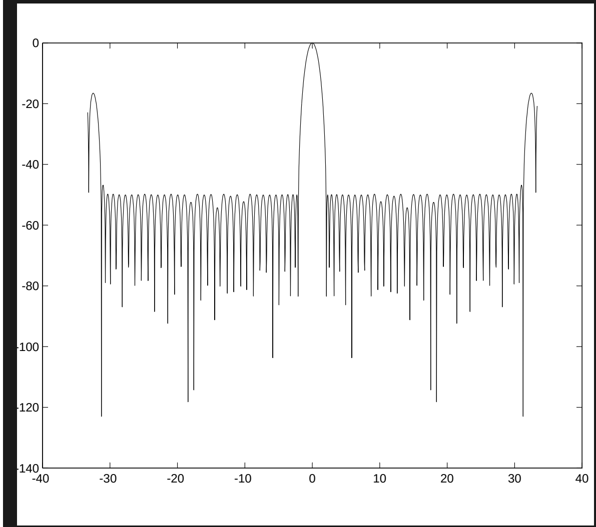
Barcode Mask as Stepping Stone...

$$\text{FWHM} = 1.3 \quad \rho_{\text{iwa}} = 2 \quad \rho_{\text{owa}} = 25$$

$$\mathcal{T}_{\text{Airy}} = 39\% \quad \text{Contrast} = 10^{-5}$$

We can test this in the lab today!





A stepping stone to Checkerboard designs...

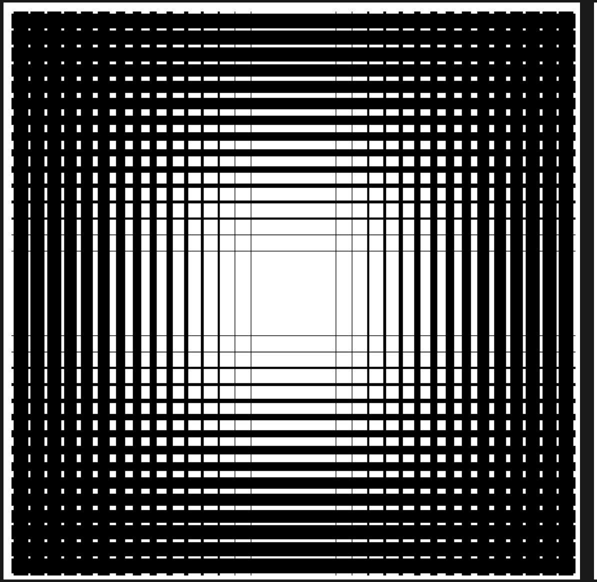
Checkerboard Mask (Two Barcodes)

Uses tensor product property of rectangular masks.

$$\mathrm{FWHM} = 0.94\sqrt{2} \quad \rho_{\mathsf{iwa}} = 2\sqrt{2} \quad \rho_{\mathsf{owa}} = 25$$

$$\mathcal{T}_{\mathsf{Airy}} = 15\% \quad \text{Contrast} = 10^{-10}$$

Uses same principal as ASA (Tensor Product of 1-D Masks)



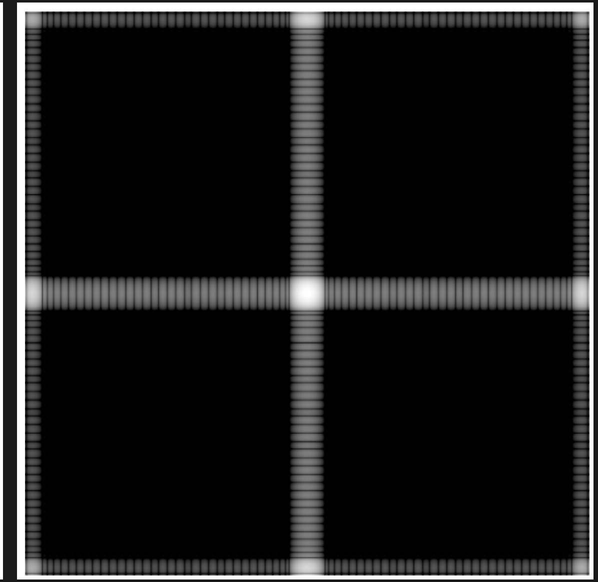
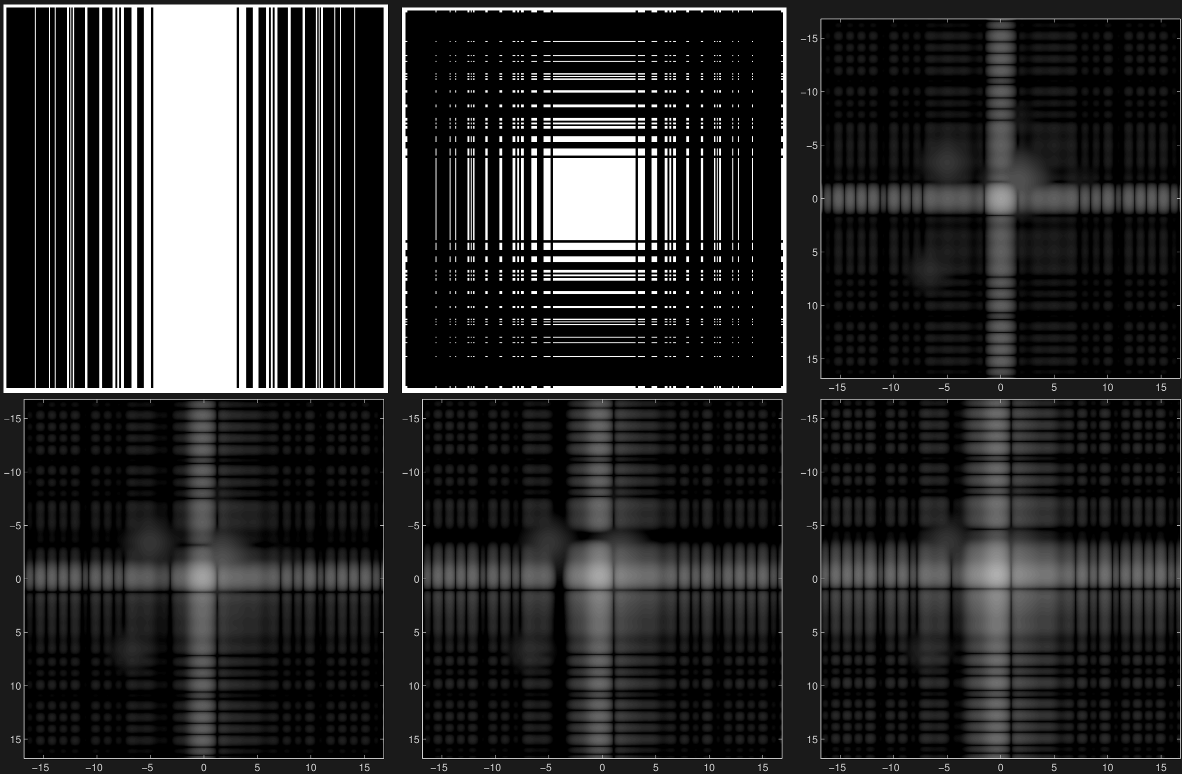


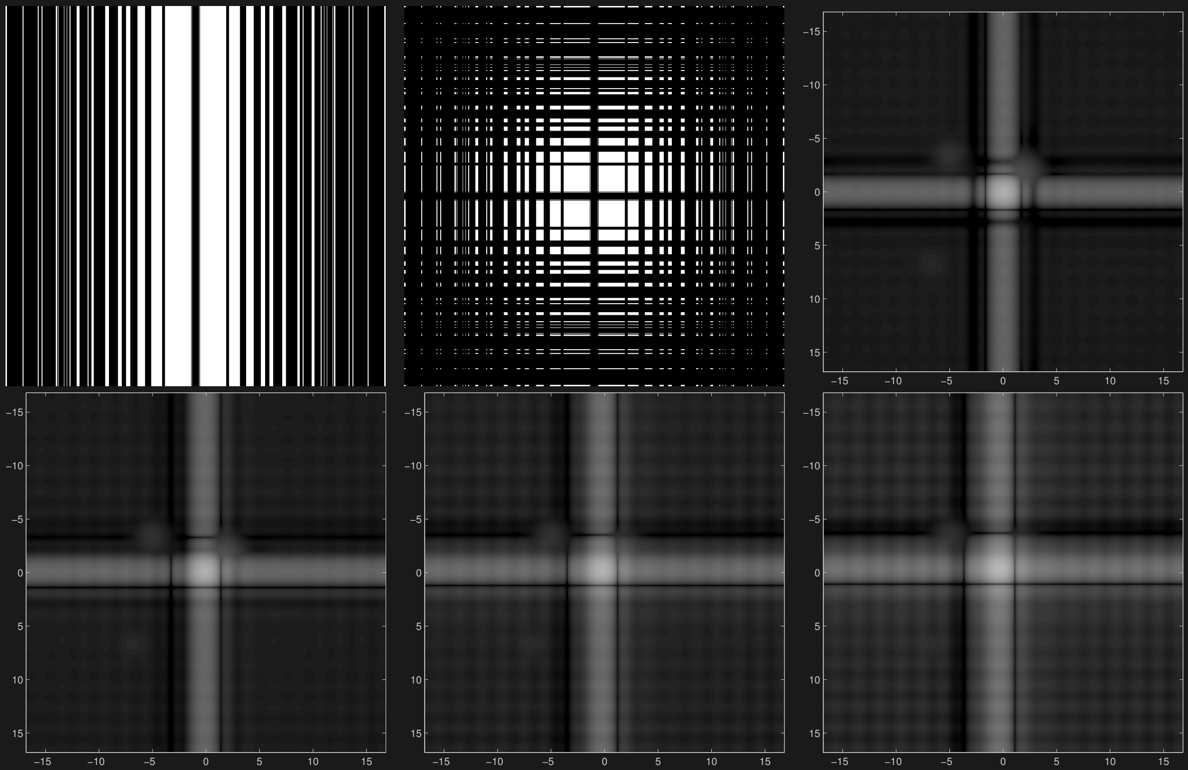
Image Plane Occulter $0.6\lambda/D$ + Checkerboard Lyot Mask

$\rho_{\text{iwa}} = 1.4\sqrt{2} \quad \rho_{\text{owa}} = 21 \quad \mathcal{T}_{\text{Airy}} = 6.2\% \quad \text{Contrast} = 10^{-10}$



... with 2% Central Obstruction

$\rho_{\text{iwa}} = 1.4 \quad \rho_{\text{owa}} = 21 \quad \mathcal{T}_{\text{Airy}} = 3.5\% \quad \text{Contrast} = 10^{-10}$



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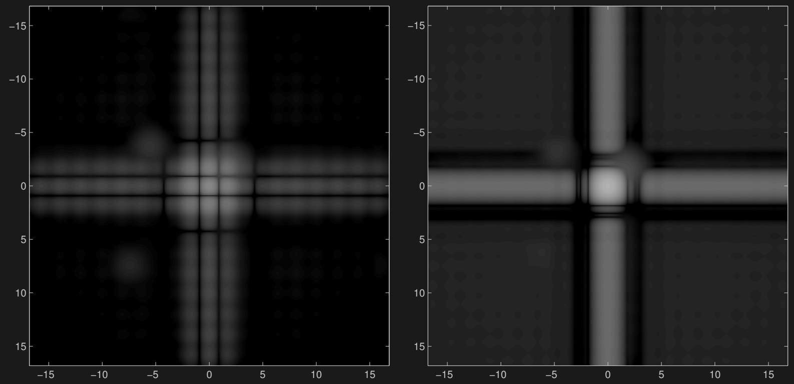
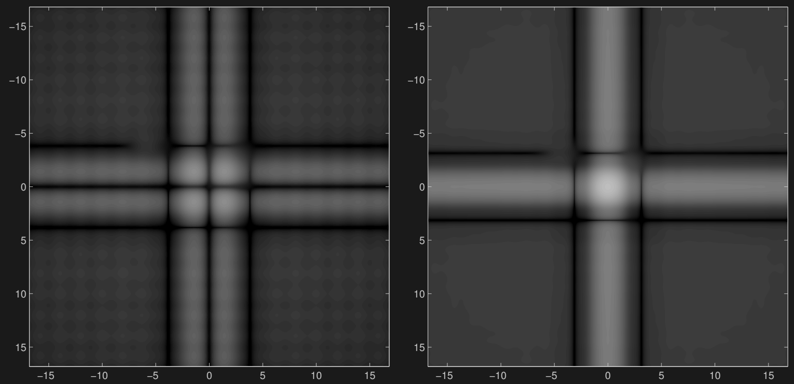
… Wavelength Dependence

$\lambda = 90\%, \quad 105\%$ of design value

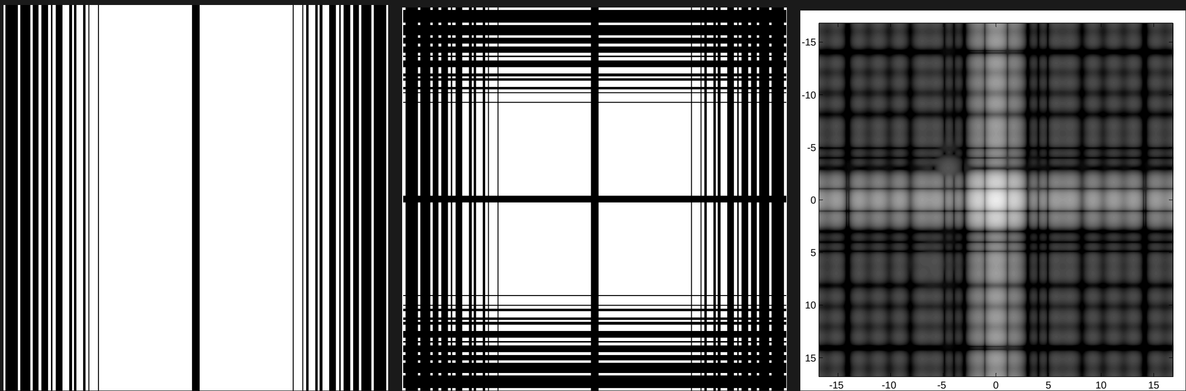


$\lambda = 85\%, \quad 110\%$ of design value



… with Higher Throughput

$$\rho_{\text{iwa}} = 4 \quad \rho_{\text{owa}} = 21 \quad \mathcal{T}_{\text{Airy}} = 40\% \quad \text{Contrast} = 10^{-10}$$

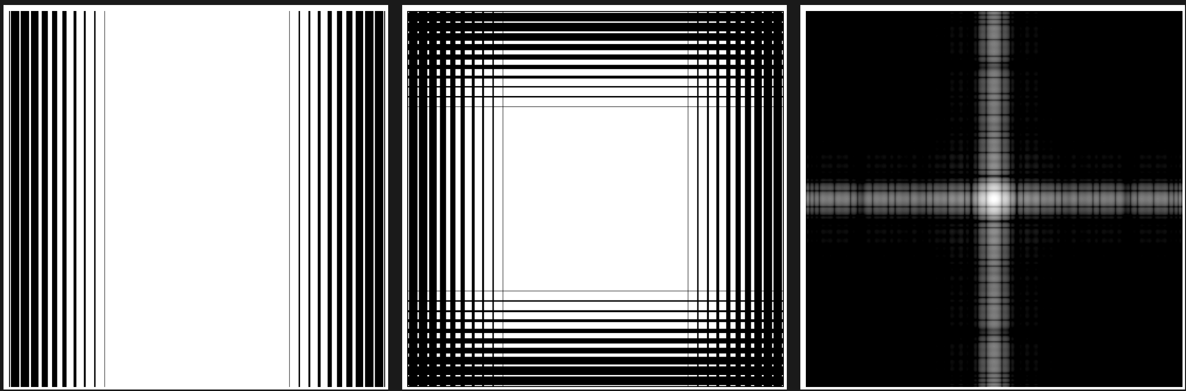


... with Neither Occulter Nor C/O

$$\text{FWHM} = 1.12\sqrt{2} \quad \rho_{\text{iwa}} = 4 \quad \rho_{\text{owa}} = 25$$

$$\mathcal{T}_{\text{Airy}} = 38\% \quad \text{Contrast} = 10^{-10}$$

Note: central lobe has side-lobes.



What may cause problems?

- Modeling Error
- Mask Manufacturing Error
- Pointing and Nonzero Stellar Size (θ^n)
- Low-Order Aberrations
- Mid-Spatial Frequencies (WFSC)
- Speckle Leak

e.g., Mask Accuracy

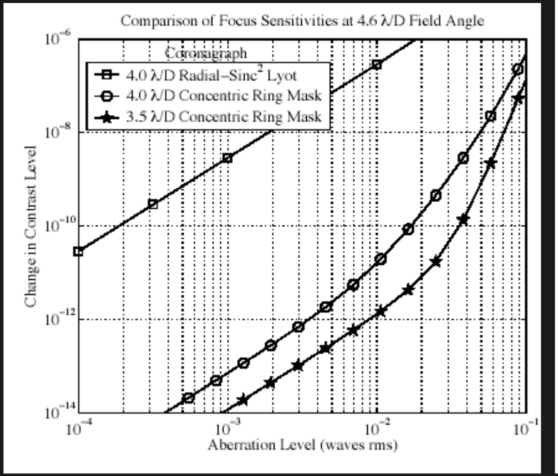
- Apodization

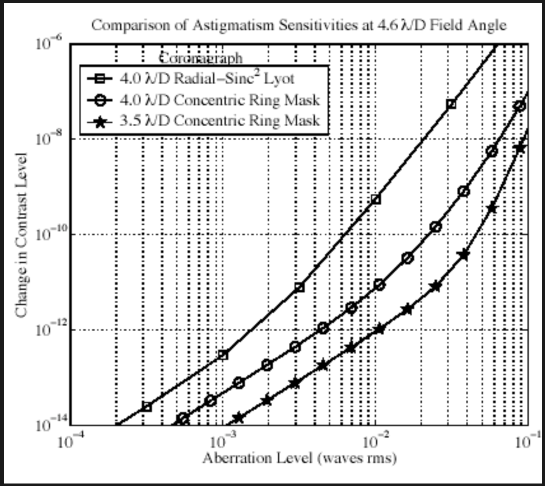
$$\epsilon \leq \frac{5 \times 10^{-6} |E_0(0,0)|}{|E_0(\xi - k_x, \zeta - k_y)|}$$

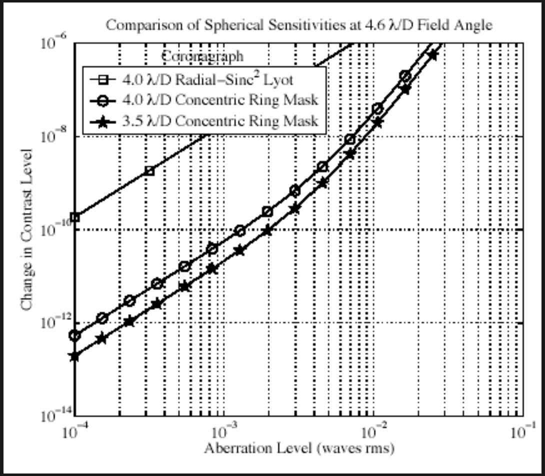
- Barcode Masks

$$\sigma \leq \frac{10^{-5} |E_0(0,0)|}{\sqrt{2N}}$$

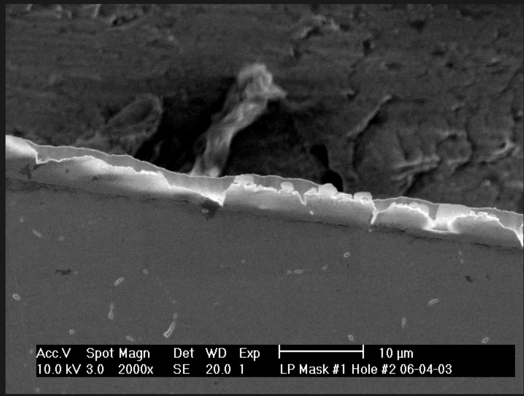
Low-Order Aberration Sensitivy

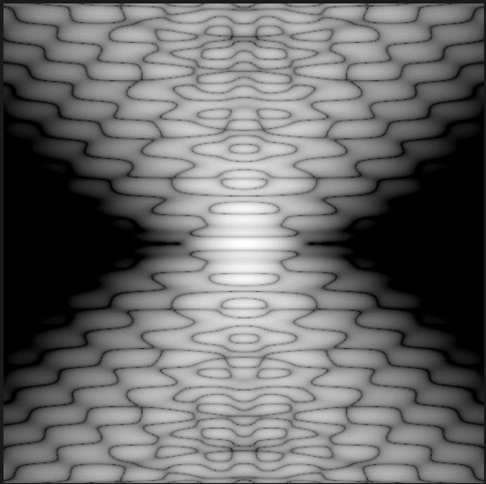


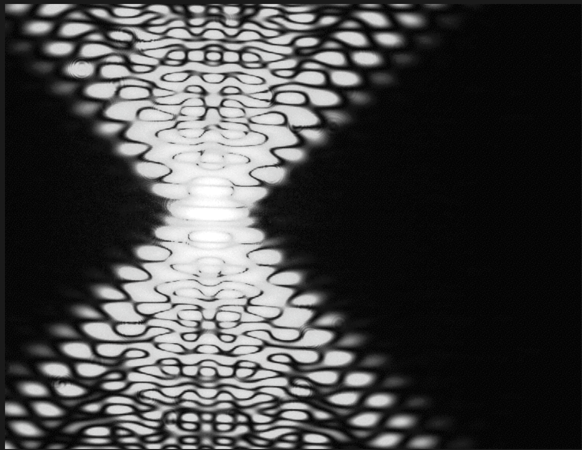




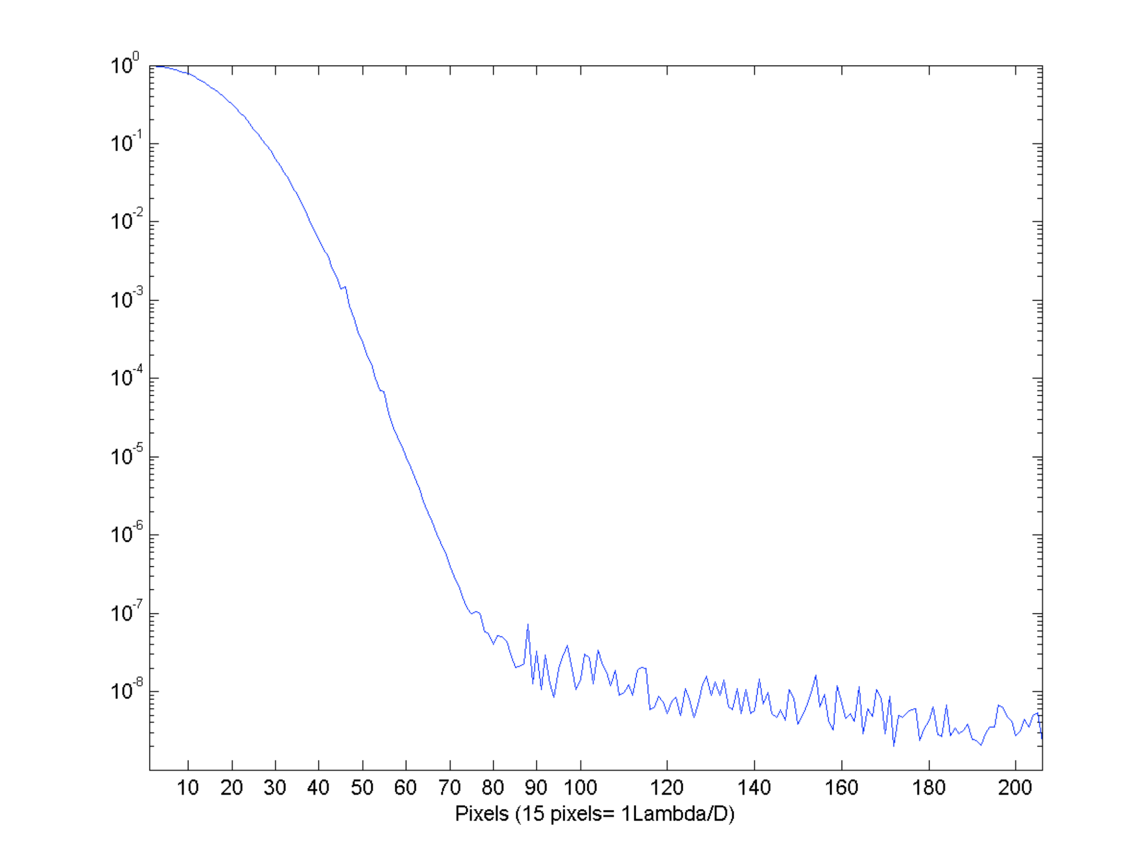
Recent Laboratory Results







Contrast Result



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