Fringe Parameter Estimation and Fringe Tracking

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Outline

- Visibility
- Fringe parameter estimation via fringe scanning
  - Phase estimation & SNR
  - Visibility estimation & SNR
- Incoherent and coherent averaging
- Estimator biases
- Fringe tracking
Visibility

- Visibility is the fundamental observable for interferometric imaging
  - Visibility is related to the object irradiance distribution via the van Cittert–Zernike theorem

- Visibility is generally complex, viz. $\Gamma = Ve^{i\phi}$
  - In optical/IR interferometry
    “visibility” generally refers to the visibility amplitude: $V = |\Gamma|$  
  - Phase is just $\text{arg}(\Gamma)$

- While object visibility can be estimated with a two-element interferometer through the atmosphere, to get true object phase requires either phase referencing (multi-beam) or closure phase (3 apertures)
Visibility is just the contrast of the spatial fringe pattern.

Or using the traditional Michelson definition:

\[ V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} \]

\[ 0 < V < 1 \]
Measuring Visibility

- Most measurement schemes involve converting the spatial pattern to a temporal pattern
  - We know how to measure the contrast of an electrical sinusoid
  - These are all variants of schemes used for phase shifting interferometry (PSI) for optical testing
    » Options
      • Step or continuous scanning
      • 4, 6, or 8 bins
      • Triangle or sawtooth waveform
  - NB: all this discussion is in context of a fringe-tracking interferometer than scans over a single interference cycle
Fringe Measurements (PTI, Keck example)

- Fringe-scanning modulation, implemented on delay line
- Sawtooth waveform to minimize number of reads per frame
- Retrace occurs during array settling time

- A, B, C, D ¼-wave intensity bins computed as
  - \( A = a - z \), \( B = b - a \), etc.
- Let \( X = A - C \), \( Y = B - D \), \( N = A + B + C + D \)

\[
\phi = \arctan\left(\frac{Y}{X}\right)
\]

\[
V^2 \propto \frac{X^2 + Y^2 - \text{bias}}{N^2}
\]
• Visibility Estimation can also be understood as a standard communication problem, aka
  – Coherent demodulation
  – Quadrature demodulation
  – Matched filtering
• Use fringe scanning to convert spatial pattern to a temporal pattern

\[ I = N(1 + V \cos(t + \phi)) \]
\[ = N + X \cos t + Y \sin t \]

\[ \phi = \tan^{-1} \frac{\hat{Y}}{\hat{X}} \]

\[ NV \propto \sqrt{\hat{X}^2 + \hat{Y}^2} \]
\[ \hat{V} \propto \frac{\sqrt{\hat{X}^2 + \hat{Y}^2}}{N} \]
4-Bin Algorithm

- Approximate sines, cosines with square waves

Slightly non-optimal, as it’s a mismatch to the proper waveform
10-20% more photons needed vs. ideal case
But minimizes number of reads
Estimating Phase

- Typically
  \[ \hat{\phi} = \tan^{-1} \frac{Y}{X} \]

\[ SNR \equiv \frac{1}{\sigma_\phi} = \sqrt{\frac{\gamma}{2} \frac{N_{tot}^2 V^2}{N_{tot} + \beta \sigma_{read\_noise}^2}} \]

\( \propto \sqrt{N} \ V, \ \text{photon-noise limited} \)
\( \propto N \ V, \ \text{read-noise limited} \)

4-bin:
\[ \frac{4}{\pi^2} \]

It’s a non-linear estimator; SNR \( \gtrsim 3 \) for proper phase estimates

Example: To obtain SNR = 5 with \( V^2 = 0.5 \)
- 125 phots, total, photon-noise limit
- 300 phots, total, with 10 electrons read noise

Improving SNR?
- General don’t average phase. Can average phasors if phase reference or closure phase – more later
Estimating Visibility

• Usually estimate $V^2$, rather than $V$, to avoid taking a square root on a noisy quantity (adds bias)

$$
\hat{V}^2 = \frac{\pi^2}{2} \frac{X^2 + Y^2 - \text{Bias}}{N^2}
$$

• Typically, inadequate SNR to get a good estimate in one sample
• Average numerator and $N$ separately

$$
\left\langle \hat{V}^2 \right\rangle = \frac{\pi^2}{2} \frac{\left\langle X^2 + Y^2 - \text{Bias} \right\rangle}{\left\langle N \right\rangle^2}
$$
SNR for $V^2$

$V^2$ is a squared quantity of Gaussian & Poisson RVs; need 4th-order statistics to compute SNR

Typically assume all noise in numerator; $N$ (in denominator) constant

**Photon-noise only**

$$SNR = \frac{1}{\sigma_{V^2}} \propto \left[ \sqrt{N}, \quad N \gg 1 \right]$$

**Read-noise only**

$$SNR = \frac{1}{\sigma_{V^2}} \propto N^2$$

NB: when photon-starved, or read noise limited, $SNR \neq \sqrt{N}$

With 2nd or higher-order estimators like for $V^2$, can get SNR dependencies steeper than $N^1$

**In general**

$$SNR = \frac{1}{\sigma_{V^2}} \propto \left( \frac{N^4}{N^2 + aN^3V^2 + b\sigma_{cds\_read\_noise}^4} \right)^{1/2}$$
Signal-to-Noise Ratio

Visibility SNR
1 sample

$\propto N^{1/2}$

$\propto N^1$

$\propto N^2$

Photons

SNR

no read noise
3 e- read noise
10 e- read noise
Coherent vs. Incoherent Averaging

Incoherent averaging (sum the magnitude squared of the fringe phasor)

- Averaging $V^2$ (strictly the numerator term) doesn’t require phase stability between samples
  - Can combine many independent estimates of $V^2$
  - At PTI, 5 spectral channels over 125 sec at 50-100 samples/sec are combined to produce a synthetic white-light $V^2$ estimate
    - Increases final SNR by ~200
    - Scatter on 25 sec points allow estimation of internal errors
  - SNR increases as $\sqrt{\text{#samples}}$
Coherent vs. Incoherent Averaging

Coherent averaging (coadding: sum the visibility phasor $N V e^{i\phi}$)
- Use a phase reference to measure the phasor rotation
- Derotate the fringe phasor ($N V e^{i\phi} \times e^{-i\phi_{\text{ref}}}$)
- Sum the fringe quadratures $X + jY$

• Compared to incoherent average
  - No advantage when samples are shot noise limited ($\text{SNR} \propto \sqrt{N}$)
    » Actually, some disadvantage due to extra biases
  - Advantage occurs when samples are photon starved
    » SNR gains faster than $\sqrt{\#\text{samples}}$

• Can also be used to increase fringe SNR to get an estimator into a linear regime
  - E.g., increase SNR to compute the arctan phase estimate

• Using a phase estimate to rotate phasors to a common angle so they can be coherently averaged is phase-referencing, a powerful technique for increasing sensitivity
Signal-to-Noise Ratio with Averaging and Coadding

Visibility SNR
10,000 total sample; 1 or 10 coadd

<table>
<thead>
<tr>
<th>SNR</th>
<th>0.0001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>photons per sample</td>
<td>2.6</td>
<td>6.3</td>
<td>15</td>
<td>45</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Blue line: no read noise
- Pink line: 10 e- read noise
Estimating Detection Bias Terms, I

- Most detectors have imperfections which must be accommodated to get good measurement accuracy
- Offsets $B_i$
  
  $N = N_{raw} - B_N$  \hspace{1cm} (from dark sky)
  
  This bias is just dark current + background

  $X = X_{raw} - B_X$  \hspace{1cm} (from dark sky)

  $Y = Y_{raw} - B_Y$  \hspace{1cm} (from dark sky)

  With a perfectly linear detector, these biases are zero
Numerator biases

\[ \text{NUM} \propto <X^2 + Y^2 - \text{bias}> \]

Photon noise

\[ <X^2 + Y^2> = k \, N \]

(can get k from slope of bias vs. N)

Counts (adc units)

Electronic gain (adc units / e^-)

+ Detector noise

\[ <X^2 + Y^2> = 4 \, k^2 \, \sigma_{\text{cds read noise}}^2 \]

(from dark sky)

Read noise variance

4 CDS reads for 4-bin algorithm

The detector-noise term dominates when read-noise limited. It also has the same noise statistics as \( V^2 \), so care must be taken in estimating it well.
Other Biases

• Atmospheric biases
  – Spatial $\langle V^2 \rangle \equiv \exp(-2\sigma^2_\phi) = \exp\left(-2.06\left(\frac{d}{r_0}\right)^{5/3}\right)$ (slow guiding)
    » Single mode fibers can eliminate most of this
  – Temporal $\langle V^2 \rangle \equiv \exp\left(-\left(\frac{T}{T_{0,2}}\right)^{5/3}\right)$
    » Some post-processing calibration possible

• Instrumental
  – Mismatched stroke vs. wavelength

• Longitudinal coherence
  – Off peak of fringe envelope
    » Narrow spectral channels for science help

NB:
The issue is not the visibility reduction, but its variability
Fringe Tracking

• What: following the interference phase - phase tracking - to stay on the central fringe to maintain coherence
  – Typically follow to ~radian
  – Maintains high duty cycle; necessary for cophasing
• [There’s also envelope tracking, which maintains centration on the fringe envelope, not discussed here]
• Issues
  – Phase measurement - already discussed
  – Sampling time
  – Phase unwrapping
  – Fringe centering
  – Atmospheric residuals
Coherence Time and Sample Spacing

- Many different definitions
  - $T_{0.2}$ - integration time during which phase fluctuations are 1 rad rms
  - $\tau_{0.2}$ - sample spacing for which phase difference = 1 rad rms
  \[ \tau_{0.2} \equiv \frac{1}{4} T_{0.2} \]
- Integration time $T < T_{0.2}$ to maintain coherence (high $V^2$)
  - rms phase fluctuations during interval $= (T/T_{0.2})^{5/6}$
- Sample spacing $t < \tau_{0.2}$ for phase continuity
  - Usually $t=T$, and this requirement dominates

\[
1 = \int df A(f)W(f,\tau)
\]

Atmospheric power spectrum: 1 or 2 aps
Window function for phase difference or variance
Phase Continuity

• Phase being measured is typically >> 2π rads
  – But arctan phase estimator −π < φ < +π
• Phase unwrapping
  – Simple
    \[ \Phi_i = 2\pi M_i + \phi_i \]
    » Chose \( M_i \) for each sample s.t. \( |\Phi_i - \Phi_{i-1}| < \pi \)
  – Better
    » Chose \( M_i \) for each sample s.t. \( |\Phi_i - \Phi_{i|-i-1}| < \pi \)
• Estimate with low pass filter or Kalman filter, matched to sample spacing, atmospheric parameters, etc.
• Sliding window can be used to improve continuity
Tracking Performance

• Typical tracker

\[ f_c \approx \frac{a}{2\pi} \] for \( f_c \ll \frac{1}{t} \)

• \( \text{rms tracking error} \approx \left( \frac{f_c}{f_{G,2}} \right)^{5/6} \)
  
  – where \( f_{G,2} \) is the two-aperture Greenwood frequency \( \propto \frac{1}{T_{0,2}} \)

• Example

  – \( T_{0,2} = 50 \text{ ms} \)
  
  – \( \tau_{0,2} = 13 \text{ ms} \)
  
  – \( f_{G,2} = 11 \text{ Hz} \)
  
  – \( T = t = 10 \text{ ms} \)
  
  – \( f_c = 5 \text{ Hz} \) (1/20\text{th} sample rate)
  
  – tracking error = 1.9 rads
Required Bandwidth

- In standard servo design, you want to optimize parameters to minimize the tracking error.
- For the interferometer, you can accurately measure the tracking error:
  - Often, you need a small enough tracking error to stay well centered on the fringe.
  - You can still co-phase even if the tracking error > 1 rad if you can feedforward to a separate delay line for the secondary channel.
Central Fringe ID

• Want to stay on the central fringe
  – Highest contrast - best SNR
  – \( V^2 \) for science also refers to central fringe
    (Typically, also use spectrometer channels with their longer coherence lengths to reduce sensitivity to tracking errors)

• How?
  – 1) Measure dependence of \( V^2 \) on phase, and move in direction of higher \( V^2 \)
  – Issues
    » \( V^2 \) estimator typically noisier than phase estimator
    » Need “wobble” -- natural or induced -- to resolve direction to move
Group Delay Estimation

- White-light fringe ≡ interference peak ≡ phases of all colors match up

\[ E = A \exp(jkx), \quad k = \frac{2\pi}{\lambda} \]

\[ x = \frac{\partial \Phi}{\partial k} \]

- Group delay estimate \( \hat{x} \) gives absolute fringe position without unwrapping errors

- Why not use all the time?
  - In the infrared, SNR for group delay worse than for phase
    - More read noise from reading additional channels
    - Incoherent group-delay estimator includes a noise term proportional to fringe envelope width \( \lambda^2/\Delta\lambda \)
Group Delay Estimation, cont.

– Usual approach to group delay in the IR
  » Use white-light phase tracking for high bandwidth control
  » Use group-delay centering at a lower bandwidth
– Different in the visible (ex: NPOI)
  » When photon count, no penalty to dispersing
  » Wide optical bandwidth reduces GD noise
    • Allows use of a coherent delay estimator which has same SNR as WL phase estimator for moderate SNRs
– Other issues
  » Atmospheric dispersion will introduce differences between the WL phase and the group delay
Conclusion

• You typically measure visibility phase and visibility amplitude by converting a spatial fringe pattern to a temporal one
  – Becomes a matched-filter problem
• You can derive SNR expressions: not everything goes at $\sqrt{N}$
  – Leads to differences between incoherent and coherent averaging
• Calibration is critical
  – Stability of biases is what frequently limits data accuracy
• Fringe tracking is implemented using the measured fringe phase
The End
Fringe derotation and stacking (coadding)

Raw phasors

Phase reference

De-rotate

(transformation matrix)

Sum (average)

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Requirements on Fringe Stabilization

Vibrations blur out the fringe - reduce fringe visibility

Need real-time control of pathlength to ~10 nm (λ/50) for high fringe visibility