MICROLENSING MAGNIFICATION CALCULATIONS WITH POINT-SOURCE AND FINITE-SOURCE EFFECTS

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MICROLENSING FOR DUMMIES
\[ A = \frac{u^2 + 2}{u \sqrt{u^2 + 4}} \]
Point-source

Point-lens

\[ A = \frac{u^2 + 2}{u \sqrt{u^2 + 4}} \]

Infinite where \( u = 0 \)
Can we assume a point-source?

- Point-source is often a reasonable assumption.
- Assumption breaks down when the source size becomes comparable to the minimum lens-source separation.
Uniform source brightness

For a source with uniform brightness, the magnification is the ratio of the area of the images to the area of the source.

\[ A = \frac{\int d^2y \, A_p(y)}{\int d^2y} \]

\[
A = \begin{cases} 
\frac{2}{\pi R_s^2} \int_{u-R_s}^{u+R_s} \frac{r^2+2}{\sqrt{r^2+4}} \arccos \frac{u^2+r^2-R_s^2}{2ur} \, dr & \text{for } u > R_s \\
\frac{2}{\pi R_s^2} \int_{R_s-u}^{R_s+u} \frac{r^2+2}{\sqrt{r^2+4}} \arccos \frac{u^2+r^2-R_s^2}{2ur} + \frac{R_s-u}{R_s^2} \sqrt{(R_s^2 - u)^2 + 4} \, dr & \text{for } u < R_s \\
\frac{2}{\pi} \left[ \left(1 + \frac{1}{R_s^2}\right) \arcsin \frac{1}{\sqrt{1 + \frac{1}{R_s^2}}} + \frac{1}{R_s} \right] & \text{for } u = R_s
\end{cases}
\]

\[ A_{max} = \sqrt{1 + \frac{4}{R_s^2}} \]

Magnification NOT infinite where \( u = 0 \)
Limb-darkening

Uniform brightness
Limb-darkening

For a source with limb-darkening, different parts of the source will be magnified differently.

\[
\begin{align*}
A &= \frac{\int d^2y \, A_p(y)}{\int d^2y} \\
A &= \frac{\int d^2y \, I(y)A_p(y)}{\int d^2y \, I(y)} \\
I(r) &= I(0)[1 - \kappa_1 Y - \kappa_2 Y^2] \\
Y &= 1 - \sqrt{1 - r^2}
\end{align*}
\]
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\[ Y = 1 - \sqrt{1 - r^2} \]

This is not trivially solved for most binary lens events!
Can we assume a point-source?

- Well it depends ...  
- Assumption breaks down when the source size becomes comparable to the minimum lens-source separation (or planetary Einstein ring)
Can we assume a point-source?

- Well it depends …
- Assumption breaks down when the source size becomes comparable to the minimum lens-source separation (or planetary Einstein ring)
- The event-rate is increasing, including that of high-magnification events!
- Need to find an optimal path for efficient computational handling of so many events with such a large parameter space
Magnification Map (ray-shooting)

- Brute-force approach
- Handles complicated including multiple lens systems very well (frequently used in quasar lensing modeling)
- Can be computationally expensive – particularly where orbital motion is included!
Application to complicated combinations of stellar densities
Magnification Map (ray-shooting)

- The image-centered approach
- Use point-source model except when source or image is close to caustic
- Shoot rays from point-source image centers
- Include partial images where disk crosses a caustic
- Polar coordinate system greatly reduces the number of needed grid points compared with Cartesian system
Light curve calculation tests: low magnification
Light curve calculation tests: high magnification
Stokes/Green Theorem

- Perform contour integration in the image plane
- Stokes' theorem generalizes integration theorems in vector calculus

\[ \oint_{\partial I} L \cdot dx = \iint_I dS (\nabla \wedge L) \cdot n \]

- Green's theorem is a special case in two dimensions

\[ \oint_{\partial I} (L_1 \, dx_1 + L_2 \, dx_2) = \iint_I dx_1 \, dx_2 \left( \frac{\partial L_2}{\partial x_1} - \frac{\partial L_1}{\partial x_2} \right) \]
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- Fast for uniform sources
Stokes/Green Theorem

- Calculate area of a microlensing image by a contour integral on the image boundary
- Sample the source boundary by approximating as a polygon
- Invert the lens equation to find image boundaries
- Re-order the points in the sample
- Approximate the contour integral using the sample
- Model limb-darkening by using rings of constant brightness
Stokes/Green Theorem

  (Application to microlensing)
  (Hybridization with inverse ray shooting)
  (Adaptive grid search)
  (Advanced contour integration)
(Advanced contour integration)
An Efficient and Robust Algorithm


(Hexagon-cell Magnification)
Map-Making (workhorse):
- Shoot rays from a narrow annulus on the image plane - reduce the overhead by orders of magnitude
- On the source plane, a combination of pixels and rays, with enhanced speed while preserving accuracy

Loop-Linking (backup) combines contour integration and ray-shooting
Hexadecapole

- Uses 13 point “grid” in the source plane
- Several times slower than point-source calculations but orders of magnitude faster than finite-source calculations
- Requires more than two source radii away from caustic
- Best when combined with other methods
- Particularly useful where planetary orbital motion is included
Hexadecapole

$$A_{\text{finite}}(\rho, x_0, y_0) \equiv \frac{\int_0^\rho dw \int_0^{2\pi} d\eta A(w, \eta)}{\pi \rho^2}$$

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Hexadecapole

\[ A_{w,+} \equiv \frac{1}{4} \sum_{j=0}^{3} A \left[ w \cos \left( \phi + j \frac{\pi}{2} \right), w \sin \left( \phi + j \frac{\pi}{2} \right) \right] - A_0 \]

\[ = A_2 w^2 + \frac{(A_{40} + A_{44})(1 + \cos^2 2\phi) + A_{42} \sin^2 2\phi}{4} w^4, \]

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caustic

crossing

cusp

Monopole
Quadrupole
Hexadecapole
Finite Source
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Practical implementation in Hands-on session after lunch

(Subo Dong & Jan Skowron)