Goals of this talk:

1. Learn to think in terms of wavelets.
2. Learn how to calculate the interference of wavefronts for any optical system.
3. Learn how to separate astrophysical from instrumental effects.

Note: Direct combination of wavefronts (homodyne detection) is discussed here, i.e. $\lambda < 10 \, \mu m$; see radio for heterodyne detection.
Summation of wavelets

Born and Wolf (7th edition, p. 428) define the wavelet summation integral as the Fourier-transform relation between amplitude in the pupil $A_{in}(x,y)$ and amplitude in the focal plane $A_{out}(u,v)$.

Image amplitude = Sum of wavelet amplitudes

$$A_{out}(u,v) = \frac{1}{\lambda f} \int A_{in}(x,y)e^{-ik(xu+yv)/f} \, dx \, dy$$

where

$$|A(x,y)|^2 = \text{energy} / \text{area} = \text{Intensity}$$

Simplify: (1) 2D→1D; (2) coef. = 1; (3) u/f = \theta = \text{angle in focal plane.}
Derive single-telescope response to point source

\[
A_{\text{tel}}(\theta) = \sum_{\text{waves}} = \int e^{i (\text{phase} \pm x)} \, dx
\]

\[
= \int_{-D/2}^{D/2} e^{i \left(\frac{2\pi D}{\lambda} \theta \right)} \, dx
\]

\[
= 2\pi i \theta \left[ e^{i \frac{\pi D}{\lambda} \theta} - e^{-i \frac{\pi D}{\lambda} \theta} \right] / D
\]

\[
= \frac{\sin \left(\frac{\pi D}{\lambda} \theta \right)}{\left(\frac{\pi D}{\lambda} \theta \right)}
\]

\[
I_{\text{tel}}(\theta) = |A_{\text{tel}}|^2 = \left[\frac{\sin \left(\frac{\pi D}{\lambda} \theta \right)}{\frac{\pi D}{\lambda} \theta} \right]^2
\]

1st zero. \( I_{\text{tel}}(\theta_{\text{tel}}) = 0 \quad \text{when} \quad \theta_{\text{tel}} = \lambda / D \)

Circular aperture. \( \int_{\text{circle}} \Rightarrow I_{\text{tel}}(\theta) = \left[\frac{2J_1 \left(\frac{\pi D}{\lambda} \theta \right)}{\frac{\pi D}{\lambda} \theta} \right]^2 \theta_{\text{tel}} = 1.22 \lambda / D \)
Single telescope again

\[
A_{tel}(\theta) = \int_{-D/2}^{+D/2} e^{i(2\pi x \theta/\lambda + \phi)} \, dx \quad / \quad I_{tel} = \frac{\sin(\pi \theta D/\lambda)}{\pi \theta D/\lambda} e^{i}
\]

I_{tel} is unchanged.

\[
A_{tel}(\theta) = \int_{-D/2}^{+D/2} e^{i(2\pi (x+\theta_0) \theta/\lambda)} \, dx \quad / \quad I_{tel} = \frac{\sin(\pi (\theta-\theta_0) D/\lambda)}{\pi (\theta-\theta_0) D/\lambda}
\]

I_{tel} is shifted to center at \( \theta_0 \).

\[
A_{tel}(\theta) = \left[ \int_{0}^{+\pi/2} e^{i(2\pi x \theta/\lambda)} \, dx + \int_{-\pi/2}^{0} e^{i(2\pi x \theta/\lambda - \pi/2)} \, dx \right] / I
\]

\[
I_{tel}(\theta) = \begin{cases} 2 & \text{speckles} 
\end{cases}
\]

i.e., 2 speckles.
Derive interferometer (2-tel.) response

$$A_{\text{int}}(\theta) = \sum_{\text{wavelets}} e^{i(\text{phase at } x)} dx$$

$$= \left[ \int_{- B + \frac{1}{2} D}^{+ B + \frac{1}{2} D} + \int_{- B + \frac{1}{2} D}^{- B - \frac{1}{2} D} \right] / 2D$$

$$= \frac{\sin (\pi \frac{\theta D}{\lambda}) \cos (\pi \frac{\theta B}{\lambda})}{\pi \frac{\theta D}{\lambda}}$$

$$I_{\text{int}}(\theta) = I_{\text{tel}}(\theta) \cdot \frac{1}{2} \left[ 1 + \cos \left( \frac{2\pi \theta B}{\lambda} \right) \right]$$

1st zero: $$I_{\text{int}}(\Theta_{\text{int}}) = 0$$ when $$\Theta_{\text{int}} = \frac{\lambda}{2B} = \text{width of fringe}.$$  

No. of fringes in packet: $$\frac{2\Theta_{\text{tel}}}{2\Theta_{\text{int}}} = \frac{1.22 \lambda / D}{0.50 \lambda / B} = 2.44 \frac{B}{D}$$
Derive binary-star response

Amplitude:

\[ A_{\text{int}}(\theta) = \sum_{\text{wavelets}} = \int e^{i(\text{phase at } x)} \, dx \]

\[ = \left[ \int_{-\frac{1}{2}B-\frac{1}{2}D}^{\frac{1}{2}B+\frac{1}{2}D} + \int_{-\frac{1}{2}B+\frac{1}{2}D}^{\frac{1}{2}B-\frac{1}{2}D} \right] / 2D \]

\[ = \frac{\sin \left( \frac{\pi D}{\lambda} \right)}{\pi D/\lambda} \cdot \cos \left( \frac{\pi B}{\lambda} \right) \]

Intensity:

\[ I_{\text{int}}(\theta) = I_{\text{tel}}(\theta) \cdot \frac{1}{2} \left[ 1 + \cos \left( \frac{2 \pi B}{\lambda} \right) \right] \]

1st zero:

\[ I_{\text{int}}(\Theta_{\text{int}}) = 0 \quad \text{when} \quad \Theta_{\text{int}} = \frac{\lambda}{2B} = \text{width of fringe.} \]

Number of fringes per packet:

\[ \text{(no. fringes)} = \frac{2 \Theta_{\text{tel}}}{\frac{2 \Theta_{\text{int}}}{\frac{2 \Theta_{\text{tel}}}{\frac{\lambda}{\Theta_{\text{int}}}}} = \frac{1.22 \lambda/D}{0.50 \lambda/B} = 2.44 \frac{B}{D} \]
Binary star interferograms

Model interferogram for a binary star, with well-separated fringe packets.

Observed interferogram of a very wide-spaced binary. CCD detector, no filter, IOTA interferometer, 1996 data.
Derive uniform disk response

\[ \text{intensity} \quad I_{\text{usb}}(\theta) = \sum_{\text{disk}} \left( \text{intensities} \right) = \int I_{\text{rad}}(\theta - \theta_x) \cdot d\theta_x \cdot d\theta_y \cdot \frac{d\theta}{d\theta_y} \]

\[ = \int I_{\text{rad}}(\theta) \cdot \frac{1}{2} \left[ 1 + \cos \left( \frac{2\pi (\theta - \theta_x) \theta}{B} \right) \right] \cdot \frac{d\theta_x}{\theta_x} \]

\[ \approx I_{\text{rad}}(\theta) \cdot \frac{1}{2} \left[ 1 + \left( \frac{\sin \frac{\pi B \theta_{\text{disk}}}{\lambda}}{\pi B \theta_{\text{disk}}/\lambda} \right) \cdot \cos \left( \frac{2\pi \theta}{\lambda} \right) \right] \]

\[ \text{visibility} \quad V_{\text{usb}} = \frac{\sin \left( \frac{\pi B \theta_{\text{disk}}}{\lambda} \right)}{\pi B \theta_{\text{disk}}/\lambda} \quad \text{square disk.} \]

\[ V_{\text{ub}} = \frac{2 I_{\text{rad}}(\pi B \theta_{\text{disk}}/\lambda)}{\pi B \theta_{\text{disk}}/\lambda} \quad \text{round disk.} \]

\[ \text{for zero} \quad V_{\text{ub}} = 0 \quad \text{when} \quad \theta_{\text{disk}} = 1.22 \frac{\lambda}{B} \quad \text{B} = 1.22 \frac{\lambda}{\theta_{\text{disk}}} \]

\[ \text{phase} \quad \phi = \begin{cases} \theta & \text{inside odd lobes} \\ \pi & \text{inside even lobes} \end{cases} \]

\[ \text{example 1} \quad \text{see next vu-graph from Burnt Wolf.} \]

\[ \text{example 2} \quad \begin{cases} \text{SIM pocket demonstration card} \end{cases} \]

\[ D = 0.07 \text{mm} \quad \text{so} \quad \theta_{\text{tel}} = 1.22 \frac{\lambda}{D} \approx 2000.7 \approx \text{sun, moon.} \]

\[ B = 0.25 \text{mm} \quad \text{so} \quad \theta_{\text{int}} = \frac{2}{AB} \approx 200.7 \approx \text{Mars,} \text{ Jupiter at 17 inches.} \]

\[ (\# \text{fringes in pocket}) = 2.44 \frac{B}{D} \approx 8. \]
Uniform disk: interferograms
Van Cittert-Zernike theorem

Beam pattern on sky.

Think like a radio astronomer. The antenna pattern is considered to be projected out from the receiver horn into antenna array onto the sky. As you move the antenna, or change the phase at an array element, the pattern sweeps across the sky. The received signal is the convolution of the moving pattern and the sources in the sky. A sinusoidal pattern picks out the Fourier component at that spacing of fringes.

van Cittert-Zernike theorem (1935, 1938)

\[
\mathcal{N}_2(\mathbf{B}) = \frac{\int_{\text{fov}} I(\mathbf{a}) \cdot e^{-i \mathbf{k} \cdot \mathbf{B} \cdot \mathbf{a}} \, d\mathbf{a}}{\int_{\text{fov}} I(\mathbf{a}) \, d\mathbf{a}}
\]

degree of coherence \( |\mu| = V = \text{visibility} \); phase = \( \arg(\mu) \).

inverse relation \( I(\mathbf{a}) \, d\mathbf{a} = \int_{\text{all } \mathbf{B}} \mu(\mathbf{B}) \cdot e^{i \mathbf{k} \cdot \mathbf{B} \cdot \mathbf{a}} \, d\mathbf{B} \)
Michelson’s stellar interferometer

Suppose we decouple the collecting apertures at B from the telescope feed apertures at $B_0$.

The coherance is measured by $B_0$.

The display pattern is set by $B_0$.

$$I_{int}(\theta) = I_{tel}(\theta) \cdot \frac{1}{2} \left[ 1 + \left( \frac{\sin \frac{\pi B \theta_{ink}}{\lambda}}{\pi B \theta_{ink}/\lambda} \right) \cos \left( 2\pi \frac{\theta B_{tel}}{\lambda} \right) \right]$$

-envelope vs $\theta$

-degree of modulation indp. of $\theta$

-modulation vs $\theta$ with period indp. of $B$ and $\theta_{ink}$

-magnification. So can make $B_0$ any convenient value. Michelson used $B_0 = 1.14$ so the fringe width is $\theta_{int} = \frac{\lambda}{2B_0} = 0.045$ arcsec.

Assume his eye had $\theta_{eye} = 1.22 \frac{2}{5} \text{mm} = 25\text{ arcsec}$.

Magnify $\theta_{int}$ to match $\theta_{eye}$ with eyepiece $M$,

$$M = \frac{\theta_{eye}}{\theta_{int}} = \frac{25}{0.045} \approx 600.$$ 

Tilt plate gives angle motion, 4 superposes images, i.e., makes wavefronts parallel at entrance pupil.

Wedge plates give variable thickness, to compensate for tilt plate’s thickness, i.e., makes all color wavefronts arrive at same time as in other beam.
Image-plane interferometer

\[ x_{\text{star}} = B \cdot \sin \theta_{\text{star}} \]

\[ \theta_{\text{star}} \]

\[ B \]

\[ x_{\text{delay}} \]

Phase difference between beams

\[ \phi = \frac{2\pi}{\lambda} (x_{\text{delay}} - x_{\text{star}}) \]

\[ I_{\text{int}}(\theta) = I_{\text{tot}}(\theta) \cdot \frac{1}{2} \left[ 1 + V \cdot \cos \frac{2\pi}{\lambda} (\theta B_0 + x_{\text{delay}} - x_{\text{star}}) \right] \]

\[ \text{envelope vs } \theta \]

\[ \text{visib. of star} \]

\[ \text{fringe modulation} \]

\[ \text{fringe phase or position} \]
Pupil-plane interferometer

Phase difference between beams
\[
\phi = \frac{2\pi}{\lambda} (X_{\text{delay}} - X_{\text{star}}) + \frac{\pi}{2}
\]

conservation of energy
at lossless beamsplitter

\( I_{\text{int}}(t) = I_{\text{tel}}(\theta) \cdot \frac{1}{2} \left[ 1 \pm \cos \frac{2\pi}{\lambda} (X_{\text{delay}} - X_{\text{star}}) \right] \)

Note: \( B_0 \) here is zero; use time modulation \( X(t) - X_{\text{star}} = \nu \)
and 1 pixel each for \( I_{\pm} \).

Spatially display each wavelength segment of \( I_{\text{int}} \), with delay \( \approx \) few \( \lambda \).
Colors in interferogram
Nulling

Nulling interferometer (Bracewell)

Star at $x=0$,
$I=0$

Planet at $x=0$,
$I=I_{\text{planet}}$

Stellar interferometer (Michelson)

Star at $x=0$,
$I=I_{\text{star}}$

Planet at $x=0$,
$I=I_{\text{planet}}$
Theta$^2$ nulling

Assume that an ideal achromatic null can be arranged. Then the intensity is

$$I_{\pm} = |e^{i k \theta r} \pm e^{-i k \theta r}|^2$$

$$= 2 \left[ 1 \pm \cos 2k \theta r \right]$$

The complementary outputs are the bright null fringes, resp.

Near the null, the intensity is quadratic

$$I_{\pm} (\theta) \approx 2 \left[ 1 - \left(1 - \frac{(2k \theta r)^2}{2!} + \cdots \right) \right]$$

$$= (2k \theta r)^2 - \cdots$$

The diagram includes a layout of an interferometer nulling setup with labels for aperture nulls, interferometer nulls, and detectors with spatial filters.
Theta$^4$ nulling

The amplitude (electric vector) in a standard null is

$$A_- = e^{i k \theta} - e^{-i k \theta} = 2i \cdot \sin(k \theta) .$$

If we could cancel this amplitude with one of opposite sign, we could make a very wide null.

The amplitude from all 4 apertures is, assuming the outer ones have relative weight $\epsilon$:

$$A_- = \epsilon e^{i k \theta'} + e^{i k \theta} - e^{-i k \theta} - (\epsilon e^{-i k \theta'})$$

$$= 2i \cdot \left\{ \sin(k \theta) - \epsilon \sin(k \theta') \right\}$$

where we have added exactly $\pi$ retardation to flip the sign.

$$A_- = k \theta (1 - \epsilon r') - \frac{(k \theta)^3 (r^3 - \epsilon r'^3)}{3!} + \ldots$$

$$= 0 \quad \text{if} \quad \epsilon = \frac{r}{r'}$$

Then $A_- = -\frac{(k \theta r)^3}{3!} \cdot (1 - \frac{r'}{r}) + \ldots$

Try using 2 more apertures:

So 2 extra mirrors, or petals, will work, provided that phase control exists.
Multiplexing in the image plane

Use minimum redundancy array at combiner lens.
Get different spatial frequency for each baseline.

\[ |\text{FFT (fringe pattern)}|^2 = \text{power spectral density} \]
Multiplexing in the pupil plane

Use different delay-line speeds.
Mix beams & get different time frequencies in each.

\[ |\text{FFT (each time sequence)}|^2 = \text{power} \]

\[ V_{12} \quad V_{23} \quad V_{13} \]

FOV. The schemes above have a FOV \( \approx \Theta_{e1} \), i.e., small, because output pupils \( \neq \) scaled input pupils.
Golden rule

Output pupil must be a scaled version of input pupil in order to obtain a wide field of view.
Pupil densification
Instrumental effects: 1

**Bandpass.** If rectangular bandpass of width $\Delta \sigma$ (where $\sigma = \frac{1}{\lambda}$), then

$$V_{\text{bandpass}} = \frac{\sin \pi \cdot x \cdot \Delta \sigma}{\pi \cdot x \cdot \Delta \sigma}$$

where $x = X_{\text{delay}} - X_{\text{star}}$.

**Wavefront tilt.** If wavefronts are tilted by angle $\alpha$, then

$$V_{\text{tilt}} = \frac{\sin \pi \cdot D \cdot \alpha / \lambda}{\pi \cdot D \cdot \alpha / \lambda} \text{ or } \frac{2 J_1(\pi D \alpha / \lambda)}{\pi D \alpha / \lambda}$$

For $V_{\text{tilt}} > 0.90$ need $\alpha < 0.3 \lambda / D$.

**Relative intensity.** If the relay optics and/or beam combiner have intensity ratio $\rho$, then

$$V_{\text{rel-int.}} = \frac{2}{\rho \cdot \alpha^2 + \rho^{-\alpha^2}}$$
Instrumental effects: 2

Non-flatness of surfaces.

If the wavefronts have rms perturbations \( \delta \), then
\[
V_{\text{surfaces}} = e^{-(2\pi \delta / \lambda)^2}.
\]

If there are \( N \) surfaces of \( \delta_0 \) each, then
\[
\delta = N^{1/2} \delta_0.
\]

If \( \delta = \lambda / 20 \) is rms of each beam, then
\[
V(\lambda/20) \approx e^{-(\pi/10)^2} \approx 0.90.
\]

Shear. No effect, unless beams no longer overlap.

Different telescope diameters. See ref. int. calc.
Filter and interferogram shapes

K-band filter transmission

K-band interferogram
Measuring visibility

\( \lambda/4 \) steps. Change \( z \) by \( (\lambda/4) \{ 0, 1, 2, 3 \} \), measure \( I(z) \), calculate \( V \sim \frac{0-2 + 1-3}{0+1 + 2+3} \).  

many \( \lambda \) sweep. Change \( x = \pi z \) and repeat in triangle wave. 
- fit theoretical wave packet to time data;
- calculate \( \text{IFFT} \{ z \} \) ratio high freq. to low;
- wavelet analysis.

dispersed (channel) spectrum. Allow atmosphere to give few \( \lambda \) path variation of \( x \)
calculate peak-to-valley variations at each \( \lambda \).
Strehl: 1

Strehl ratio is approximately
\[ S = e^{-\varphi^2} \]
where \( \varphi \) is the rms phase error across a wavefront.

Observed visibility is the product of 3 terms:
\[ V_{\text{observed}} = S_{\text{atmos}} S_{\text{instrum}} V_{\text{object}} \]

Instrumental Strehl ratio is the product of many terms:
\[ S_{\text{instrum}} = S_{\text{servo}} S_{\text{flat}} S_{\text{align}} S_{\text{diffraction}} S_{\text{flux}} S_{\text{overlap}} S_{\text{vibration}} S_{\text{window}} S_{\text{polarization}} \]
Atmospheric variance, with tip-tilt removed by a servo system with bandwidth $v/\pi D$, is

$$\varphi^2 = (0.134 + 0.096)(D/r_0)^{5/3}(\lambda_0/\lambda)^2$$

Wavefront flatness variance from mirror surfaces is

$$\varphi^2 = \varphi_1^2 + \ldots + \varphi_n^2$$

Mirrors are often specified in terms of surface peak-to-valley where an empirical relation is

$$PV = 5.5 \text{ RMS}$$
Polarization and visibility

(a) order = \( y \times e \).

(b) order = \( e \times y \).

S - P phase shift
\( \lambda = 406, 450, 506, 604, 706, 826 \) nm
from measured \( \{ n, k \} \)
silver over chrome

Phase shift (deg)

Incidence angle (deg)
Visibility reduction factor

Beam 1: \( \vec{A}_1 = (A_x, A_y)_1 = a_1 e^{ikz} (1, e^{i\phi_2}) \)

Beam 2: \( \vec{A}_2 = (A_x, A_y)_2 = a_2 e^{ik(z+\delta)} (1, e^{i\phi_2}) \)

Combined: \( I = |\vec{A}_1 + \vec{A}_2|^2 \)

\[
I = \bar{I} \left[ \frac{1}{2} + \frac{\cos (kz + \phi)}{a_1^2 + a_2^2} \cdot \cos (kl + \phi) \cdot |\cos \frac{\phi}{2}| \right]
\]

where \( \phi = \phi_2 - \phi_1 \) = relative phase shift between 2 beams.

\% In unpolarized light, the measured visibility can be permanently degraded by purely instrumental polarization effects, in the amount

\[ |\cos \frac{\phi}{2}| \]

which is a function of wavelength only.
Single-mode fiber optics

Core:  
- Core index $n_1$, radius $a$ ($n_1 \approx 1.48$, $a \approx 2 \mu m$).
- Cladding index $n_2$, radius $b$ ($n_2 \approx 1.46$, $b \approx 60 \mu m$).
- Incident/exit cone $\sin \varphi = (n_1^2 - n_2^2)^{1/2} \leq NA \leq 14^\circ$.

Dispersion:  
- Intermodal $= 0$ for SM fiber.
- Material, waveguide balance these to get zero.

Waveguide parameter:  
$V = \frac{2\pi}{\lambda_0} a (n_1^2 - n_2^2)^{1/2} > 10 \Rightarrow$ geometric optics
$< 10 \Rightarrow$ wave optics.

Coupling:  
$0 \rightarrow 0 \rightarrow 1$  
$2 \approx$ coupling length $\approx 3 \mu m$ typically.

Ftg. couplings:  
- Twist & melt (fused)
- Polish & mate (polished).

Cutoff $\lambda_c$:  
For $\lambda > \lambda_c$, the fiber is single-mode.
$\lambda_c = \frac{2\pi a}{3.848 (n_1^2 - n_2^2)^{1/2}}$.  
If $NA \approx 0.24$, $\lambda_c \approx 0.6 \mu m$.

Eendue:  
$AE = \pi a^2 \pi \gamma^2 = \pi a^2 \pi (NA)^2 \approx (1.202 \lambda_c)^2 \approx \lambda^2$.

Transmission:  
Loss $< 1 \text{ dB/km}$ for silica & fluoride.

Integrated optics:  
Fibers on a chip.
Injecting starlight into a fiber

- The optimal $f/d$ is the one that maximizes the overlap integral
- Maximum possible efficiency: $\rho_{\text{max}} = 78\%$ (but less if the pupil has a central obstruction $\alpha$)

- Efficiency set by the overlap integral: in the focal plane: $\rho = \left| \int E_{\text{tel}}^* E_{\text{fibre}} \right|^2$
  $\Rightarrow$ the field amplitudes must match
Electric vector control
Integrated optics: 1

Figure 1. Description of the IONICLODA three-way beam combiner. Three inputs are split with three "Y" junctions to provide a pairwise beam combination with another an. of three couplers.

Figure 2. Schematic description of the IONIC bench (see text for details).
Integrated optics: 2